Minimization of the Amount of Straight Reinforcement in Composites

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Publication date:
1988

Document Version
Publisher's PDF, also known as Version of record

Link to publication from Aalborg University

Citation for published version (APA):
Rathkjen, A. (1988). *Minimization of the Amount of Straight Reinforcement in Composites*. Institute of Building Technology and Structural Engineering. R : Institut for Bygningsteknik, Aalborg Universitet Vol. R8812 No. 1988.1

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1. INTRODUCTION

When a weak matrix has to be reinforced in order to increase its strength the most efficient way to do so is to place the reinforcement in such a way that it is directed along the principal stress-trajectories. Since the directions of these trajectories usually change from point to point in a structure the result will be a curvilinear arrangement of the reinforcement. Though efficient this may be quite an expensive way of producing a strong composite structure.

More often the reinforcement is placed along fixed directions chosen in advance for some practical reason. The question of determining the smallest amount of reinforcement sufficient to carry a given load while placed in fixed directions has been given some attention in connection with reinforced concrete. A solution for reinforced concrete in plane stress was given by Nielsen in 1969, (Nielsen [69.01] [84.01]). Since plain concrete is a brittle material in tension it is assumed that the concrete cracks so that it is in a state of uniaxial compression when tensile reinforcement is required. The determination of the direction of these cracks plays a dominant role in the development of formulae for minimum reinforcement in concrete.

The development here is not based on the assumption that the matrix cracks. It may be used in connection with any criterion of failure for the matrix.

In section 2 the basic principles are reviewed and some examples with orthogonal reinforcement in two directions are given in section 3. When a composite is reinforced in two orthogonal directions only, the shear stress corresponding to these two directions has to be carried by the matrix alone. When the shear stress cannot be carried by the matrix alone reinforcement in at least a third direction has to be supplied. An example of this is given in section 4.

The basic principles in section 2 are quite general, while the examples in sections 3 and 4 cover plane stress only.

2. BASIC PRINCIPLES

In a composite material reinforced with fibres, threads or bars in N directions the total stress σ_{kl} in the composite is expressed by the stress m_{kl} in the matrix, the volume fraction φ^n for the n'th direction and the extra stress t^n_{kl} in the reinforcement in that direction as

$$\sigma_{kl} = m_{kl} + \sum_{n=1}^{N} \varphi^n t^n_{kl} \tag{1}$$

The extra stress t_{kl} in the reinforcement is the difference between the total stress in the reinforcement and the matrix stress m_{kl}. When the reinforcement consists of fibres, threads or bars it is assumed that the extra stress may be approximated with a uniaxial tension or compression t in the direction a_m of the reinforcement so that

$$t_{kl}^n = t^n a_k^n a_l^n \tag{2}$$

for the n'th direction.

With the notation $(\varphi t a_k a_l)^n = \varphi^n t^n a_k^n a_l^n$ the relation between the stress in composite, matrix and reinforcement is

$$\sigma_{kl} = m_{kl} + \sum_{n=1}^{N} (\varphi t a_k a_l)^n \tag{3}$$

The extra stress in the reinforcement is limited by

$$-t_c \leqq t \leqq t_t \tag{4}$$

where t_c and t_t are maximum extra stresses in compression and tension respectively. In order to utilize the reinforcement as effectively as possible the extra stress has to have one of its maximum values or no reinforcement at all should be provided in the direction under consideration.

The total amount of reinforcement in a unit volume of composite is

$$\varphi = \sum_{n=1}^{N} \varphi^n \tag{5}$$

Since, however, high strength is expensive, it is more appropriate to minimize a measure φt defined as

$$\varphi t = \sum_{n=1}^{N} (\varphi t_M)^n \tag{6}$$

where t_M equals either t_t or t_c as the state of stress dictates.

By using (3) the measure of reinforcement φt is expressed in terms of the composite stress σ_{kl} and the matrix stress m_{kl}. The composite stress σ_{kl} is determined from stress analysis and regarded as constant at each point of the structure while the matrix stress m_{kl} has to be determined in such a way that φt attains a minimum. The components of the matrix stress are restricted by the failure criterion

$$f(m_{kl}) = 0 \tag{7}$$

for the matrix material and again the composite is utilized most effectively when both matrix and reinforcement fails. Defining a function

$$F = \varphi t(m_{kl}) - \lambda f(m_{kl}) \tag{8}$$

where λ is a Lagrangian multiplier, minimum reinforcement is found by minimizing F.

Since equation (8) consists of no more than 6 (in plane stress 3) scalar equations, only reinforcement in 6 (3) directions can be minimized. If more directions are used the amount of reinforcement in some directions have to be determined from other considerations.

3. REINFORCEMENT IN TWO DIRECTIONS

In a composite reinforced in two orthogonal directions x and y the stress-components in plane stress are

$$\sigma_{xx} = m_{xx} + \varphi_x t_x$$

$$\sigma_{yy} = m_{yy} + \varphi_y t_y \qquad (9)$$

$$\sigma_{xy} = m_{xy}$$

Three values of $\varphi_x t_x$, viz. $\varphi_x t_{tx}$, 0 , $-\varphi_x t_{cx}$, have to be investigated together with three values of $\varphi_y t_y$, viz. $\varphi_y t_{ty}$, 0 , $-\varphi_y t_{cy}$, giving a total of nine instances as shown in table 1.

	$\varphi_x t_x$	$\varphi_y t_y$
1	$\varphi_x t_{tx}$	$\varphi_y t_{ty}$
2	$-\varphi_x t_{cx}$	$-\varphi_y t_{cy}$
3	$\varphi_x t_{tx}$	$-\varphi_y t_{cy}$
4	$-\varphi_x t_{cx}$	$\varphi_y t_{ty}$
5	$\varphi_x t_{tx}$	0
6	0	$\varphi_y t_{ty}$
7	$-\varphi_x t_{cx}$	0
8	0	$-\varphi_y t_{cy}$
9	0	0

Table 1.

In the following subsections these investigations are performed in connection with different matrix failure criteria.

3.1 QUADRATIC FAILURE CRITERION

A matrix failure criterion that is a complete polynomial of the second degree may be given as

$$I^2/CT - II/S^2 + (C-T)I/CT - 1 = 0 \qquad (10)$$

where I and II are the two first principal invariants of the stress tensor and C, T and S are the strengths in uniaxial compression, in uniaxial tension, and in pure shear, respectively. The criterion can be used for isotropic materials only. As shown in [86.01] the criterion may be written as

$$\frac{4S^2-CT}{4CTS^2}\left(m_{xx}+m_{yy}+2\frac{(C-T)S^2}{4S^2-CT}\right)^2+\frac{(m_{xx}-m_{yy})^2}{4S^2}$$

$$+\frac{m_{xy}^{\ 2}}{S^2}-\frac{(C+T)^2S^2-C^2T^2}{(4S^2-CT)CT}=0 \qquad (11)$$

in plane stress. With

$$A^2 = \frac{S^2}{(4S^2-CT)^2}((C+T)^2S^2-C^2T^2)$$

$$Q^2 = \frac{S^2}{(4S^2-CT)CT}((C+T)^2S^2-C^2T^2) \qquad (12)$$

$$B = \frac{(C-T)S^2}{4S^2-CT}$$

this expression reduces to

$$f = Q^2(m_{xx}+m_{yy}+2B)^2 + A^2(m_{xx}-m_{yy})^2 + 4A^2m_{xy}^2 - 4A^2Q^2 = 0 \qquad (13)$$

which will be used when minimizing $F = \varphi t - \lambda f$.

1) With $\varphi_x t_x = \varphi_x t_{tx}$ and $\varphi_y t_y = \varphi_y t_{ty}$ both sets of reinforcement are in tension and we have from (9)

$$\varphi_x t_{tx} = \sigma_{xx} - m_{xx}$$

$$\varphi_y t_{ty} = \sigma_{yy} - m_{yy} \qquad (14)$$

$$m_{xy} = \sigma_{xy}$$

so that

$$F = \varphi t - \lambda f = \varphi_x t_{tx} + \varphi_y t_{ty} - \lambda f$$

$$= \sigma_{xx} - m_{xx} + \sigma_{yy} - m_{yy} - \lambda(Q^2 (m_{xx} + m_{yy} + 2B)^2$$

$$+ A^2 (m_{xx} - m_{yy})^2 + 4A^2 (\sigma_{xy}^2 - Q^2)) \qquad (15)$$

Minimum reinforcement is found for

$$\partial F/\partial m_{xx} = -1 - \lambda(2Q^2 (m_{xx} + m_{yy} + 2B) + 2A^2(m_{xx} - m_{yy})) = 0$$

$$(16)$$

$$\partial F/\partial m_{yy} = -1 - \lambda(2Q^2(m_{xx} + m_{yy} + 2B) - 2A^2(m_{xx} - m_{yy})) = 0$$

Elimination of λ gives

$$m_{xx} = m_{yy} \qquad (17)$$

and from the failure criterion (13) we find

$$m_{xx} = m_{yy} = A\sqrt{1-(\sigma_{xy}/Q)^2} - B \qquad (18)$$

so that

$$\varphi_x t_{tx} = \sigma_{xx} + B - A\sqrt{1-(\sigma_{xy}/Q)^2}$$

$$\varphi_y t_{ty} = \sigma_{yy} + B - A\sqrt{1-(\sigma_{xy})Q)^2} \qquad (19)$$

$$\lambda = 1/(4AQ\sqrt{Q^2 - \sigma_{xy}^2})$$

Since each volume fraction has to be positive, this result is valid when

$$\sigma_{xx} \geqq A\sqrt{1-(\sigma_{xy})Q)^2} - B$$

$$(20)$$

$$\sigma_{yy} \geqq A\sqrt{1-(\sigma_{xy})Q)^2} - B$$

2) With $\varphi_x t_x = -\varphi_x t_{cx}$ and $\varphi_y t_y = -\varphi_y t_{cy}$ both sets of reinforcement are in compression and equations (9) give

$$\varphi_x t_{cx} = m_{xx} - \sigma_{xx}$$
$$\varphi_y t_{cy} = m_{yy} - \sigma_{yy} \tag{21}$$
$$m_{xy} = \sigma_{xy}$$

Corresponding to 1), minimum reinforcement is found for

$$m_{xx} = m_{yy} = -A\sqrt{1-(\sigma_{xx}/Q)^2} - B \tag{22}$$

and

$$\varphi_x t_{cx} = -\sigma_{xx} - B - A\sqrt{1-(\sigma_{xy}/Q)^2}$$
$$\varphi_y t_{cy} = -\sigma_{yy} - B - A\sqrt{1-(\sigma_{xy}/Q)^2} \tag{23}$$
$$\lambda = -1/(4AQ\sqrt{Q^2-\sigma_{xy}^2})$$

The amount of reinforcement is positive when

$$\sigma_{xx} < -B - A\sqrt{1-(\sigma_{xy}/Q)^2}$$
$$\sigma_{yy} < -B - A\sqrt{1-(\sigma_{xy}/Q)^2} \tag{24}$$

3) Compression in one set of reinforcement and tension in the other is first treated with $\varphi_x t_x = \varphi_x t_{tx}$ and $\varphi_y t_y = -\varphi_y t_{cy}$. From (9)

$$\varphi_x t_{tx} = \sigma_{xx} - m_{xx}$$
$$\varphi_y t_{cy} = m_{yy} - \sigma_{yy} \tag{25}$$
$$m_{xy} = \sigma_{xy}$$

so that

$$F = \varphi t - \lambda f = \varphi_x t_{tx} + \varphi_y t_{cy} - \lambda f =$$
$$= \sigma_{xx} - m_{xx} + m_{yy} - \sigma_{yy} - \lambda(Q^2(m_{xx} + m_{yy} + 2B)^2 +$$
$$+ A^2(m_{xx} - m_{yy})^2 + 4A^2(\sigma_{xy}^2 - Q^2)) \tag{26}$$

Minimum reinforcement is found from

$$\partial F/\partial m_{xx} = -1 - \lambda(2Q^2(m_{xx} + m_{yy} + 2B) + 2A^2(m_{xx} - m_{yy})) = 0$$
$$\partial F/\partial m_{yy} = 1 - \lambda(2Q^2(m_{xx} + m_{yy} + 2B) - 2A^2(m_{xx} - m_{yy})) = 0$$
(27

giving

$$m_{xx} = + Q\sqrt{1-(\sigma_{xy}/Q)^2} - B$$
$$m_{yy} = - Q\sqrt{1-(\sigma_{xy}/Q)^2} - B \qquad (28)$$

and

$$\varphi_x t_{tx} = \sigma_{xx} + B - Q\sqrt{1-(\sigma_{xy}/Q)^2}$$
$$\varphi_y t_{cy} = - \sigma_{yy} - B - Q\sqrt{1-(\sigma_{xy}/Q)^2} \qquad (29)$$
$$\lambda = -1/(4A^2\sqrt{Q^2-\sigma_{xy}^2})$$

valid for

$$\sigma_{xx} \geqq - B + Q\sqrt{1-(\sigma_{xy}/Q)^2}$$
$$\sigma_{yy} \leqq - B - Q\sqrt{1-(\sigma_{xy}/Q)^2} \qquad (30)$$

4) The results for $\varphi_x t_x = - \varphi_x t_{cx}$ and $\varphi_y t_y = \varphi_y t_{ty}$ are

$$m_{xx} = - Q\sqrt{1-(\sigma_{xy}/Q)^2} - B$$
$$m_{yy} = + Q\sqrt{1-(\sigma_{xy})Q)^2} - B \qquad (31)$$

and

$$\varphi_x t_{cx} = - \sigma_{xx} - B - Q\sqrt{1-(\sigma_{xy})Q)^2}$$
$$\varphi_y t_{ty} = \sigma_{yy} + B - Q\sqrt{1-(\sigma_{xy}/Q)^2} \qquad (32)$$
$$\lambda = 1/(4A^2\sqrt{Q^2-\sigma_{xy}^2})$$

valid for

$$\sigma_{xx} \leqq -B - Q\sqrt{1-(\sigma_{xy}/Q)^2}$$
$$\sigma_{yy} \geqq -B + Q\sqrt{1-(\sigma_{xy}/Q)^2} \qquad (33)$$

5) With no reinforcement in one direction and tension in the other, e.g. $\varphi_x t_x = \varphi_x t_{tx}$ and $\varphi_y t_y = 0$, equations (9) give

$$\begin{aligned} \varphi_x t_{tx} &= \sigma_{xx} - m_{xx} \\ m_{yy} &= \sigma_{yy} \\ m_{xy} &= \sigma_{xy} \end{aligned} \tag{34}$$

No minimization takes place, the matrix stress m_{xx} is determined from the failure criterion (13) with $m_{yy} = \sigma_{yy}$ and $m_{xy} = \sigma_{xy}$ from (34). We find

$$m_{xx} = -B + \frac{1}{A^2 + Q^2} \left((A^2 - Q^2)(\sigma_{yy} + B) + 2A\sqrt{(A^2 + Q^2)(Q^2 - \sigma_{xy}^2) - Q^2(\sigma_{yy} + B)^2}\right)$$

$$\varphi_x t_{tx} = \sigma_{xx} + B - \frac{1}{A^2 + Q^2}\left((A^2 - Q^2)(\sigma_{yy} + B) + 2A\sqrt{(A^2 + Q^2)(Q^2 - \sigma_{xy}^2) - Q^2(\sigma_{yy} + B)^2}\right) \tag{35}$$

valid for

$$Q^2(\sigma_{xx} + \sigma_{yy} + 2B) + A^2(\sigma_{xx} - \sigma_{yy})^2 + 4A^2(\sigma_{xy}^2 - Q^2) \geqq 0 \tag{36}$$

In the other cases with reinforcement in one direction only, we have

6) $\varphi_x t_x = 0$, $\varphi_y t_y = \varphi_y t_{ty}$

$$m_{yy} = -B + \frac{1}{A^2 + Q^2}\left((A^2 - Q^2)(\sigma_{xx} + B) + 2A\sqrt{(A^2 + Q^2)(Q^2 - \sigma_{xy}^2) - Q^2(\sigma_{xx} + B)^2}\right) \tag{37}$$

$$\varphi_y t_{ty} = \sigma_{yy} - m_{yy}$$

7) $\varphi_x t_x = -\varphi_x t_{cx}$, $\varphi_y t_y = 0$

$$m_{xx} = -B + \frac{1}{A^2 + Q^2}\left((A^2 - Q^2)(\sigma_{yy} + B) - 2A\sqrt{(A^2 + Q^2)(Q^2 - \sigma_{xy}^2) - Q^2(\sigma_{yy} + B)^2}\right) \tag{38}$$

$$\varphi_x t_{cx} = m_{xx} - \sigma_{xx}$$

8) $\varphi_x t_x = 0$, $\varphi_y t_y = -\varphi_y t_{cy}$

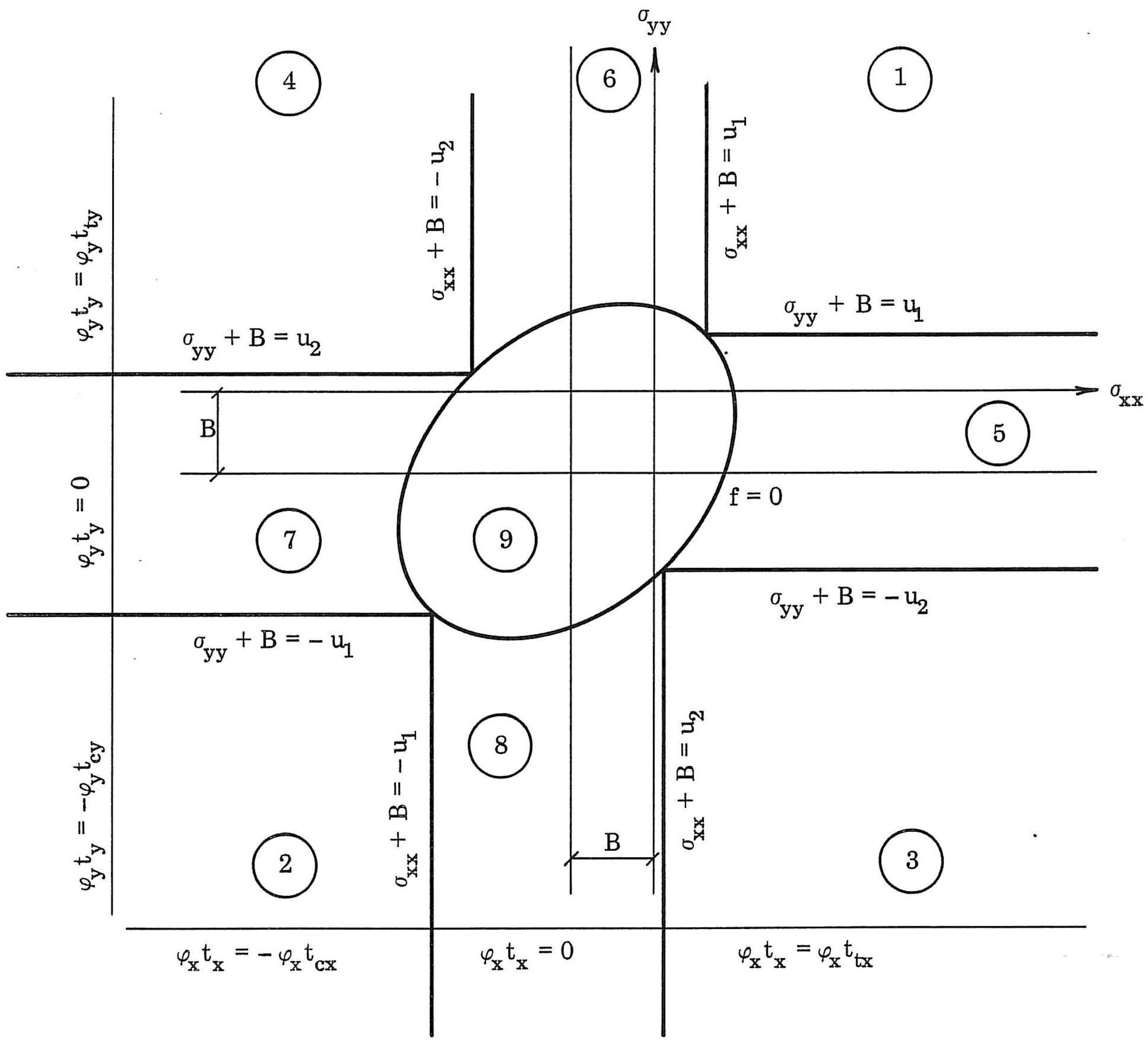
$$m_{yy} = -B + \frac{1}{A^2 + Q^2}\left((A^2 - Q^2)(\sigma_{xx} + B) - 2A\sqrt{(A^2 + Q^2)(Q^2 - \sigma_{xy}^2) - Q^2(\sigma_{xx} + B)^2}\right) \tag{39}$$

$$\varphi_y t_{cy} = m_{yy} - \sigma_{yy}$$

9) When $\varphi_x t_x = \varphi_y t_y = 0$, (9) shows that the matrix stress equals the composite stress and no reinforcement is required when

$$Q^2(\sigma_{xx} + \sigma_{yy} + 2B)^2 + A^2(\sigma_{xx} - \sigma_{yy})^2 + 4A^2\sigma_{xy}^2 - 4A^2Q^2 \leqq 0 \tag{40}$$

The results are summarized in figure 1 and table 2. Figure 1 shows a σ_{xx}, σ_{yy} – plane in a $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ – space. The plane is divided into 9 fields, each field corresponds to a particular set of values of the parameters $\varphi_x t_x$, $\varphi_y t_y$, m_{xx} and m_{yy}. Values of these parameters are given in table 2 while equations for the boundary-lines are given in figure 1. Amounts of reinforcement are computed using equations (9).



$$f(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = Q^2(\sigma_{xx} + \sigma_{yy} + 2B)^2 + A^2(\sigma_{xx} - \sigma_{yy})^2 + 4A^2\sigma_{xy}^2 - 4A^2Q^2$$

$$u_1(\sigma_{xy}) = A\sqrt{1-(\sigma_{xy}/Q)^2}$$

$$u_2(\sigma_{xy}) = Q\sqrt{1-(\sigma_{xy}/Q)^2}$$

Figure 1.

Field	$\varphi_x t_x$	$\varphi_y t_y$	$m_{xx} + B$	$m_{yy} + B$
1	$\varphi_x t_{tx}$	$\varphi_y t_{ty}$	$A\sqrt{1-(\sigma_{xy}/Q)^2}$	$A\sqrt{1-(\sigma_{xy}/Q)^2}$
2	$-\varphi_x t_{cx}$	$-\varphi_y t_{cy}$	$-A\sqrt{1-(\sigma_{xy}/Q)^2}$	$-A\sqrt{1-(\sigma_{xy}/Q)^2}$
3	$\varphi_x t_{tx}$	$-\varphi_y t_{cy}$	$Q\sqrt{1-(\sigma_{xy}/Q)^2}$	$-Q\sqrt{1-(\sigma_{xy}/Q)^2}$
4	$-\varphi_x t_{cx}$	$\varphi_y t_{ty}$	$-Q\sqrt{1-(\sigma_{xy}/Q)^2}$	$Q\sqrt{1-(\sigma_{xy}/Q)^2}$
5	$\varphi_x t_{tx}$	0	$K_1 + L_1$	$\sigma_{yy} + B$
6	0	$\varphi_y t_{ty}$	$\sigma_{xx} + B$	$K_2 + L_2$
7	$-\varphi_c t_{cx}$	0	$K_1 - L_1$	$\sigma_{yy} + B$
8	0	$-\varphi_y t_{cy}$	$\sigma_{xx} + B$	$K_2 - L_2$
9	0	0	$\sigma_{xx} + B$	$\sigma_{yy} + B$

$$K_1 = (A^2 - Q^2)(\sigma_{yy} + B)/(A^2 + Q^2)$$

$$L_1 = 2A\sqrt{(A^2 + Q^2)(Q^2 - \sigma_{xy}^2) - Q^2(\sigma_{yy} + B)^2}/(A^2 + Q^2)$$

$$K_2 = (A^2 - Q^2)(\sigma_{xx} + B)/(A^2 + Q^2)$$

$$L_2 = 2A\sqrt{(A^2 + Q^2)(Q^2 - \sigma_{xy}^2) - Q^2(\sigma_{xx} + B)^2}\ /(A^2 + Q^2)$$

Table 2.

3.2 PRINCIPAL STRESS FAILURE CRITERION

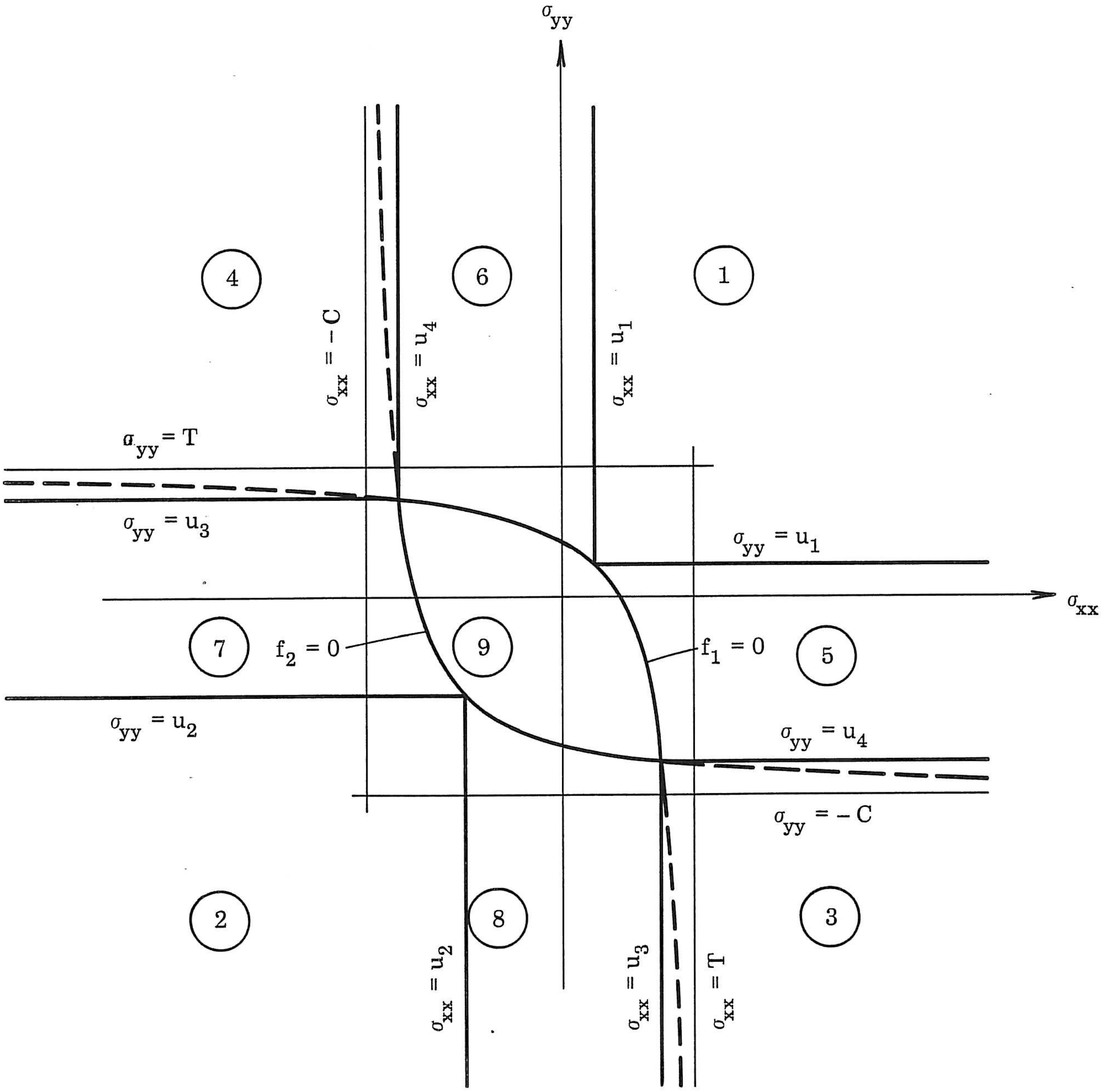
The principal stress criterion in plane stress consists of two failure functions

$$f_1 = -(m_{xx} - T)(m_{yy} - T) + \sigma_{xy}^2 = 0$$

$$f_2 = -(m_{xx} + C)(m_{yy} + C) + \sigma_{xy}^2 = 0 \qquad (41)$$

Minimum reinforcement is found following the same procedure as in subsection 3.1. The results are summarized in figure 2 and table 3, to be used together with equations (9).

In plane stress and with the tensile matrix-strength T = 0 the principal stress criterion is identical with Conlomb's modified criterion used by Nielsen ([69.01], [84.01], and the amounts of reinforcement found are the same. Nielsen did not use compressive reinforcement and consequently he did not use the fields 2, 3, 4, 7, and 8.



$$f_1(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = -(\sigma_{xx} - T)(\sigma_{yy} - T) + \sigma_{xy}^2$$

$$f_2(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = -(\sigma_{xx} + C)(\sigma_{yy} + C) + \sigma_{xy}^2$$

$$u_1(\sigma_{xy}) = T - \sqrt{\sigma_{xy}^2}$$

$$u_2(\sigma_{xy}) = -C + \sqrt{\sigma_{xy}^2}$$

$$u_3(\sigma_{xy}) = \frac{1}{2}\left(T - C + \sqrt{(T + C)^2 - 4\sigma_{xy}^2}\right)$$

$$u_4(\sigma_{xy}) = \frac{1}{2}\left(T - C - \sqrt{(T + C)^2 - 4\sigma_{xy}^2}\right)$$

Figure 2.

Field	$\varphi_x t_x$	$\varphi_y t_y$	m_{xx}	m_{yy}
1	$\varphi_x t_{tx}$	$\varphi_y t_{ty}$	$T - \sqrt{\sigma_{xy}^2}$	$T - \sqrt{\sigma_{xy}^2}$
2	$-\varphi_x t_{cx}$	$-\varphi_y t_{cy}$	$-C + \sqrt{\sigma_{xy}^2}$	$-C + \sqrt{\sigma_{xy}^2}$
3	$\varphi_x t_{tx}$	$-\varphi_y t_{cy}$	$\frac{1}{2}(T - C + \sqrt{(T+C)^2 - 4\sigma_{xy}^2})$	$\frac{1}{2}(T - C - \sqrt{(T+C)^2 - 4\sigma_{xy}^2})$
4	$-\varphi_x t_{cx}$	$\varphi_y t_{ty}$	$\frac{1}{2}(T - C - \sqrt{(T+C)^2 - 4\sigma_{xy}^2})$	$\frac{1}{2}(T - C + \sqrt{(T+C)^2 - 4\sigma_{xy}^2})$
5	$\varphi_x t_{tx}$	0	$T + \sigma_{xy}^2/(\sigma_{yy} - T)$	σ_{yy}
6	0	$\varphi_y t_{ty}$	σ_{xx}	$T + \sigma_{xy}^2/(\sigma_{xx} - T)$
7	$-\varphi_x t_{cx}$	0	$-C + \sigma_{xy}^2/(\sigma_{yy} + C)$	σ_{yy}
8	0	$-\varphi_y t_{cy}$	σ_{xx}	$-C + \sigma_{xy}^2/(\sigma_{xx} + C)$
9	0	0	σ_{xx}	σ_{yy}

Table 3.

4. REINFORCEMENT IN THREE DIRECTIONS

When a matrix is reinforced in two orthogonal directions only, the shear stress has to be carried by the matrix alone as equation (9 c) reveals. This is possible only when the shear stress does not exceed the shear capacity of the matrix material. The shear capacity is generally not equal to the shear strength of the material corresponding to the stress-state pure shear. In case of a material that fails according to the quadratic criterion, the shear capacity is Q where Q is given by the expression (12 b), and, in case of a material that fails according to the principal stress criterion the shear capacity is (T + C)/2.

If the shear stress exceeds the shear capacity at least two possibilities are open. The easy way is to increase the thickness (and the weight) of the disk thus reducing the stress to a value below the capacity, and then use the results from section 3. Another possibility is to provide the matrix with reinforcement in three directions. The three directions may be chosen arbitrarily, here two orthogonal directions, x and y, and a direction inclined an angle $\theta = 45^{\circ}$ with x and y will be considered. The arrangement is shown in figure 3.

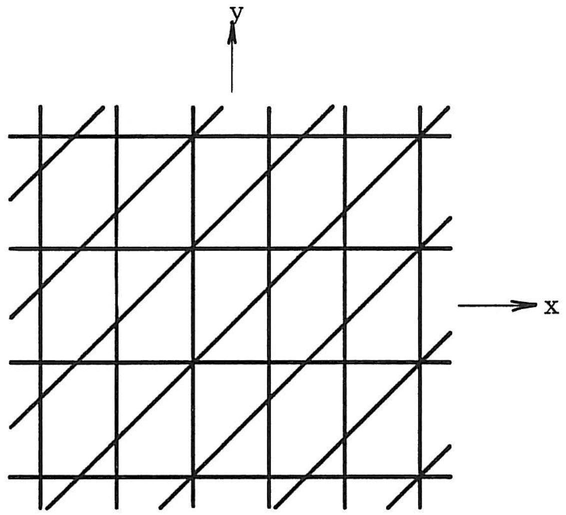


Figure 3.

From (3) the relations between composite stress, matrix stress and extra stress in the reinforcement are

$$\sigma_{xx} = m_{xx} + \sum_n (\varphi t \cos^2\theta)_n = m_{xx} + \varphi_x t_x + \frac{1}{2}\varphi_\theta t_\theta$$

$$\sigma_{yy} = m_{yy} + \sum_n (\varphi t \sin^2\theta)_n = m_{yy} + \varphi_y t_y + \frac{1}{2}\varphi_\theta t_\theta \qquad (42)$$

$$\sigma_{xy} = m_{xy} + \sum_n (\varphi t \sin\theta \cos\theta)_n = m_{xy} + \frac{1}{2}\varphi_\theta t_\theta$$

where φ_x, φ_y and φ_θ are volume fractions and t_x, t_y and t_θ are extra stress in the x, y and $\theta = 45^o$ directions.

Using the same procedure and the same quadratic matrix failure criterion as described in subsection 3.1, values of matrix stresses m_{xy}, m_{xx} and m_{yy} are found giving minimum of $\varphi t = \varphi_x t_x + \varphi_y t_y + \varphi_\theta t_\theta$. The results are summarized in figures 4 - 23 and tables 4 - 5.

Several figures are needed when a material is reinforced in three directions, each figure corresponding to a domain of values of the shear stress σ_{xy}. Also two sets of figures have to be used according as A is less than or greater than 2Q.

One set (figures 4 - 13) has to be used when $A < 2Q$ ($\Rightarrow Q/\sqrt{2} < 2Q^2/\sqrt{4Q^2 + A^2}$), the other set (figures 14 - 23) has to be used when $2Q < A$ ($\Rightarrow 2Q^2/\sqrt{4Q^2 + A^2} < Q/\sqrt{2}$).

In both cases the boundaries between the fields in the figures are determined by the functions given in table 4, and values of $\varphi_\theta t_\theta$, $\varphi_x t_x$, $\varphi_y t_y$, m_{xy}, $m_{xx} + B$ and $m_{yy} + B$ corresponding to each field are given in table 5.

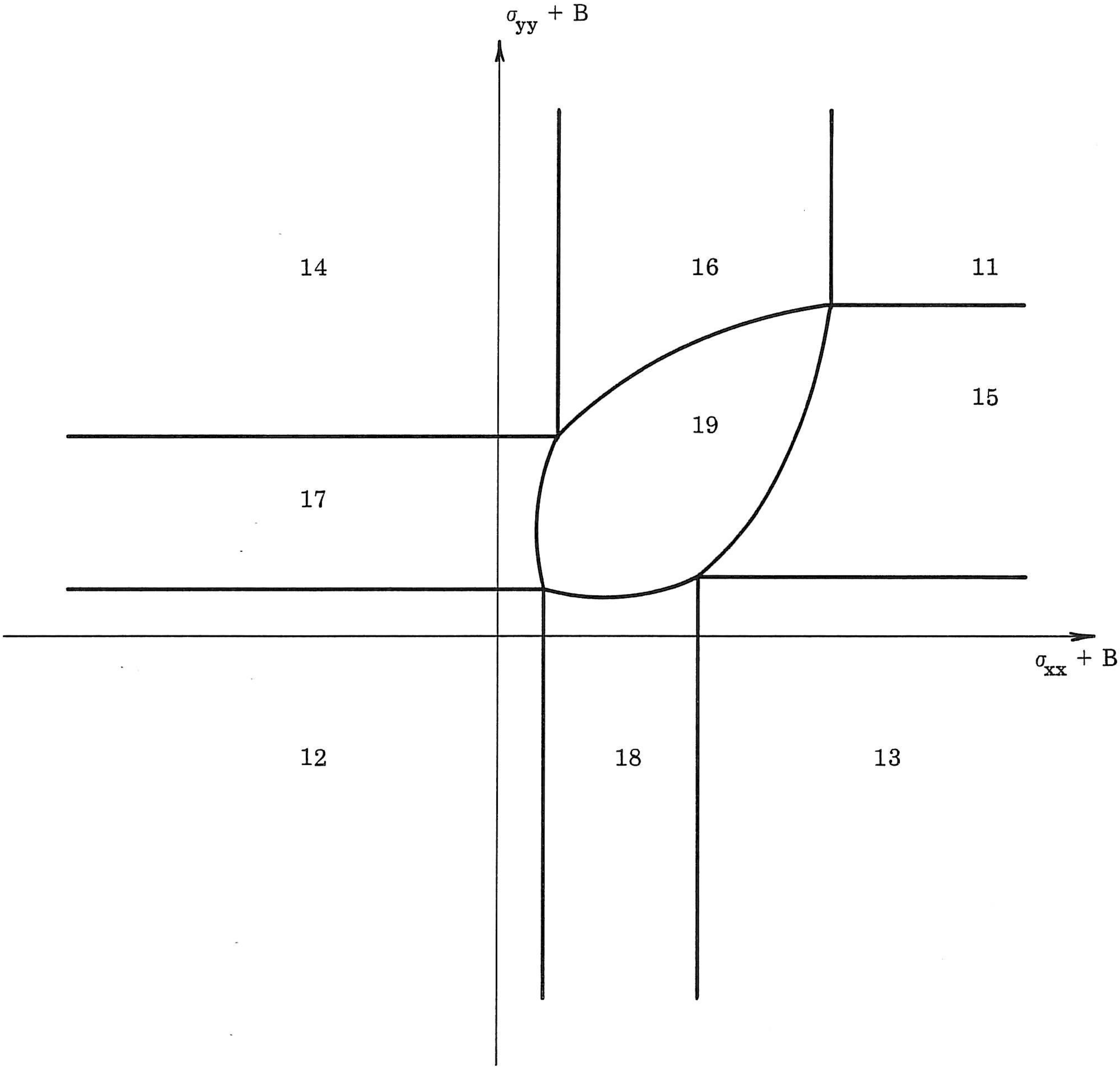
Amounts of reinforcement are computed from eqs. (42) using table 5, first φ_θ from

$$\sigma_{xy} = m_{xy} + \frac{1}{2}\varphi_\theta t_\theta \qquad (43)$$

then φ_x and φ_y from

$$\sigma_{xx} = m_{xx} + \varphi_x t_x + \frac{1}{2}\varphi_\theta t_\theta$$

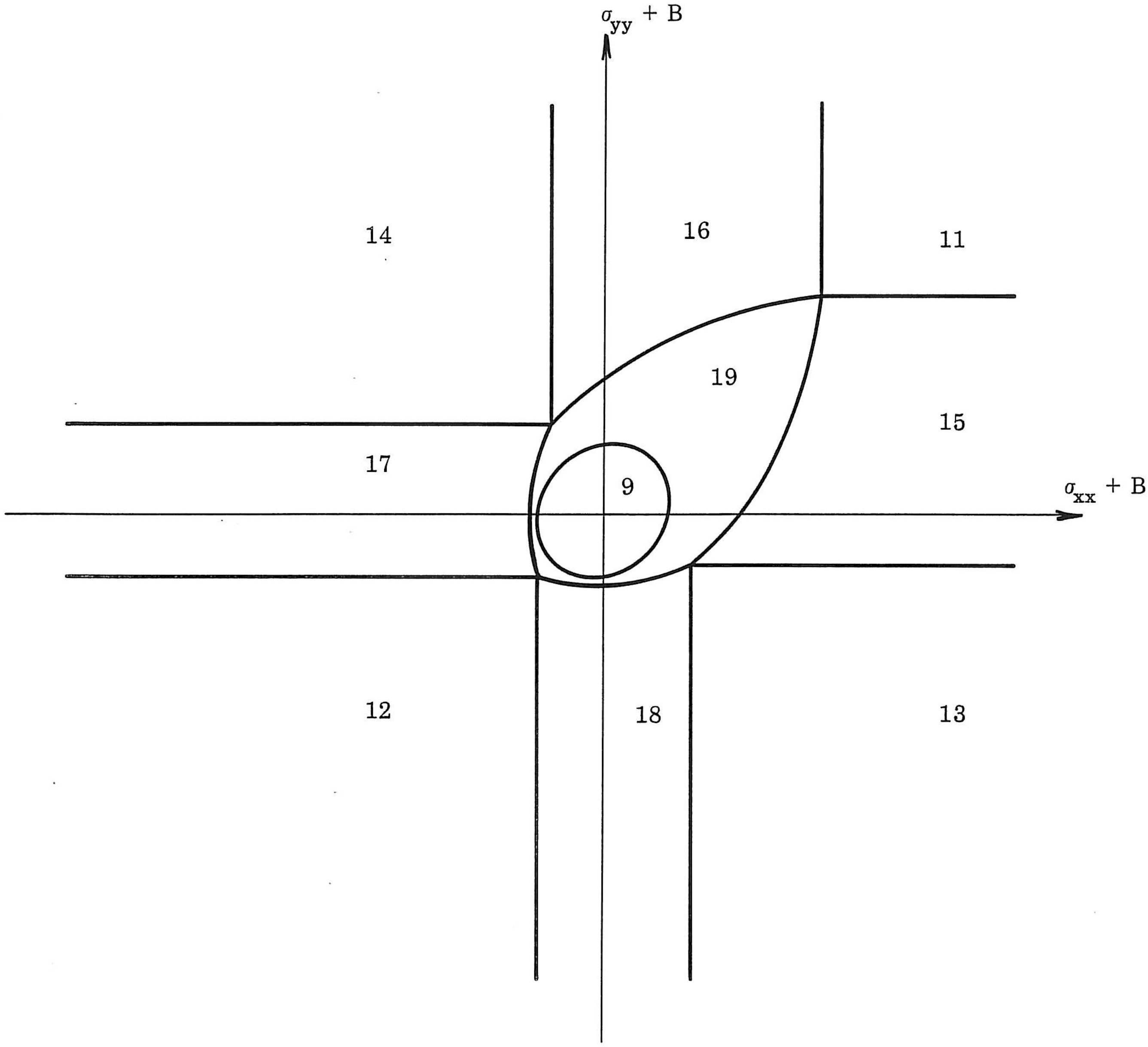
$$\sigma_{yy} = m_{yy} + \varphi_y t_y + \frac{1}{2}\varphi_\theta t_\theta \qquad (44)$$



A < 2Q

Q < σ_{xy}

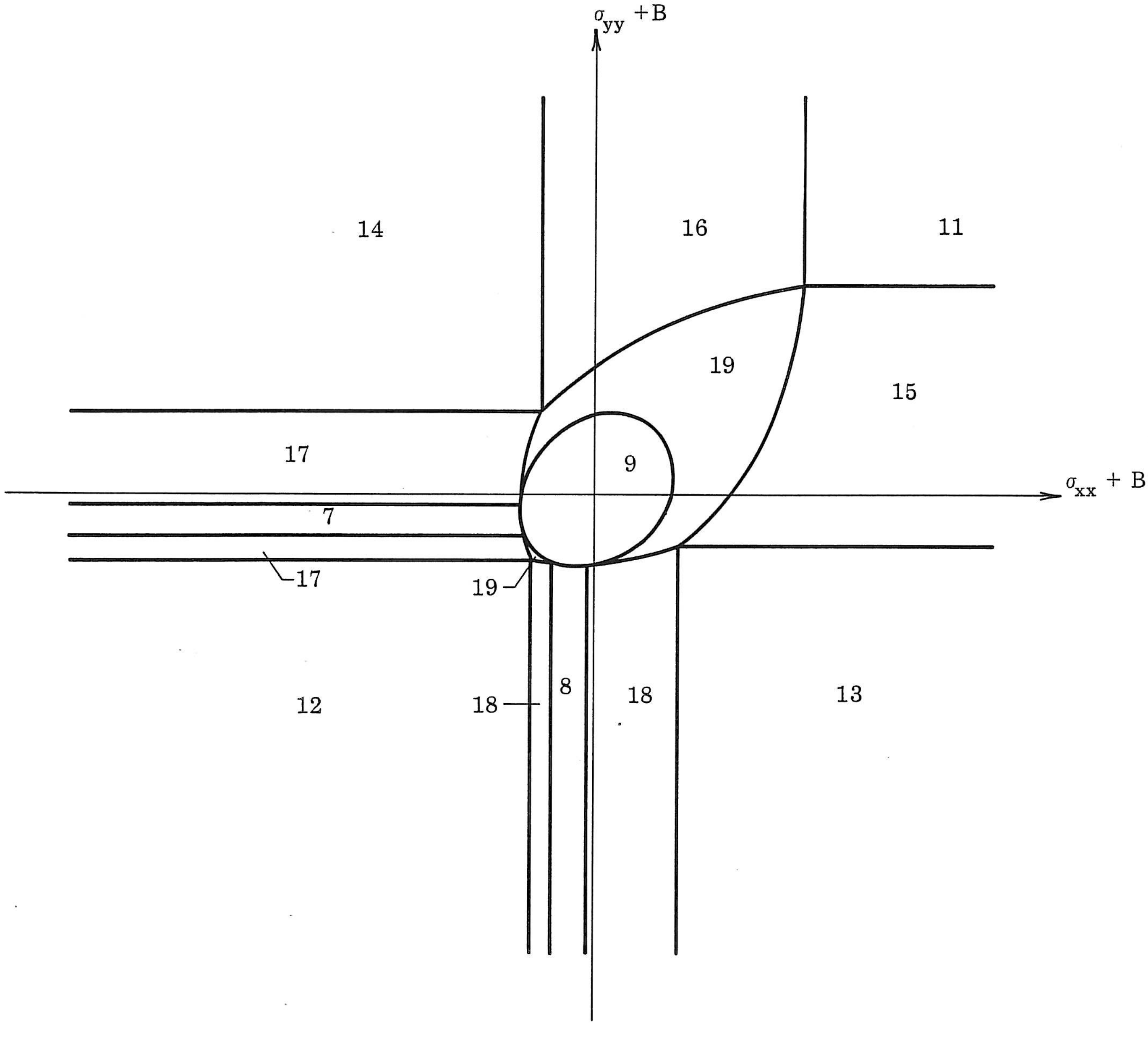
Figure 4.



A < 2Q

$Q\sqrt{4Q^2 + A^2} \,/\, \sqrt{2(2Q^2 + A^2)} < \sigma_{xy} < Q$

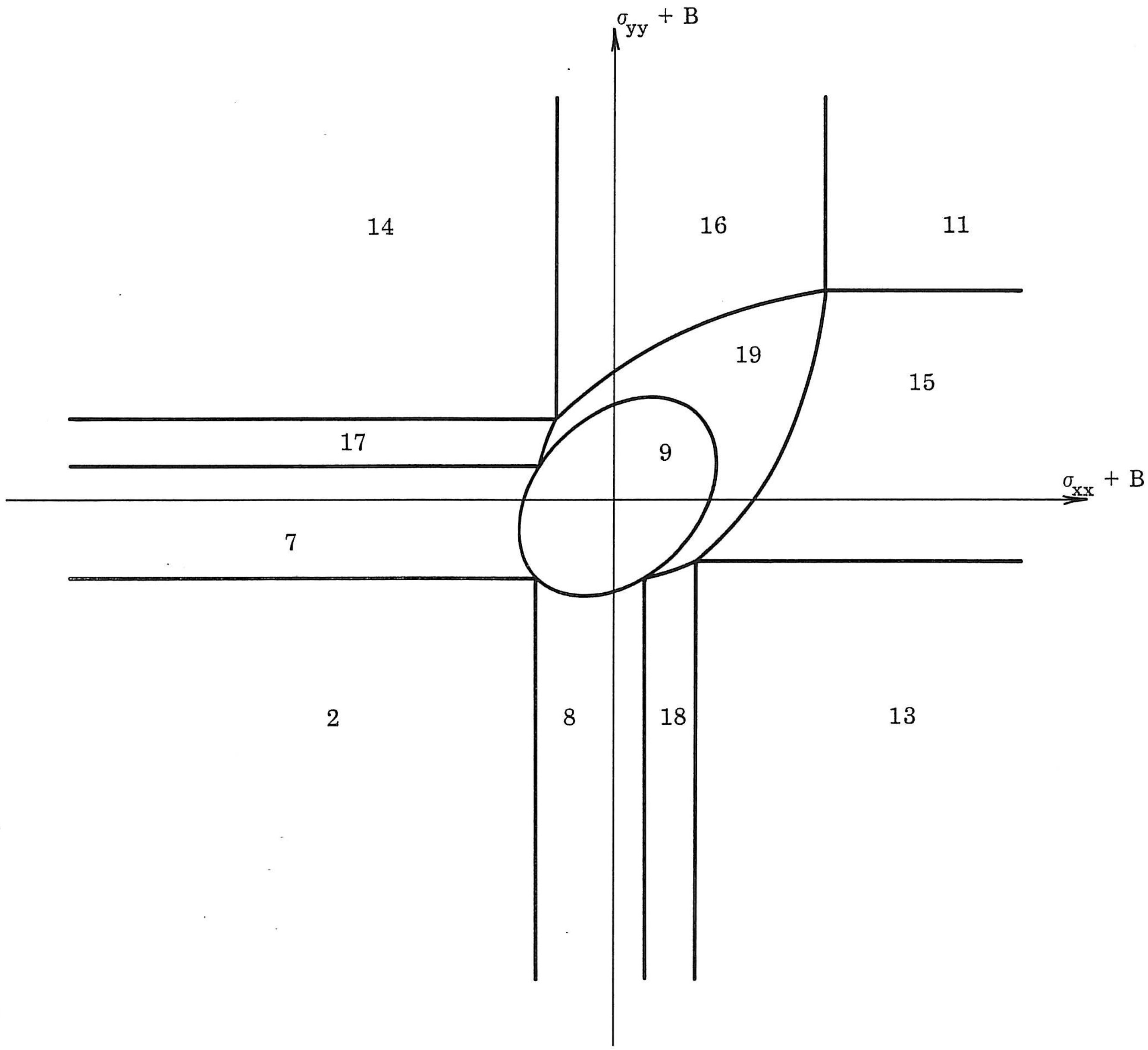
Figure 5.



A < 2Q

$2Q^2/\sqrt{4Q^2 + A^2} < \sigma_{xy} < Q\sqrt{4Q^2 + A^2}\,/\sqrt{2(2Q^2 + A^2)}$

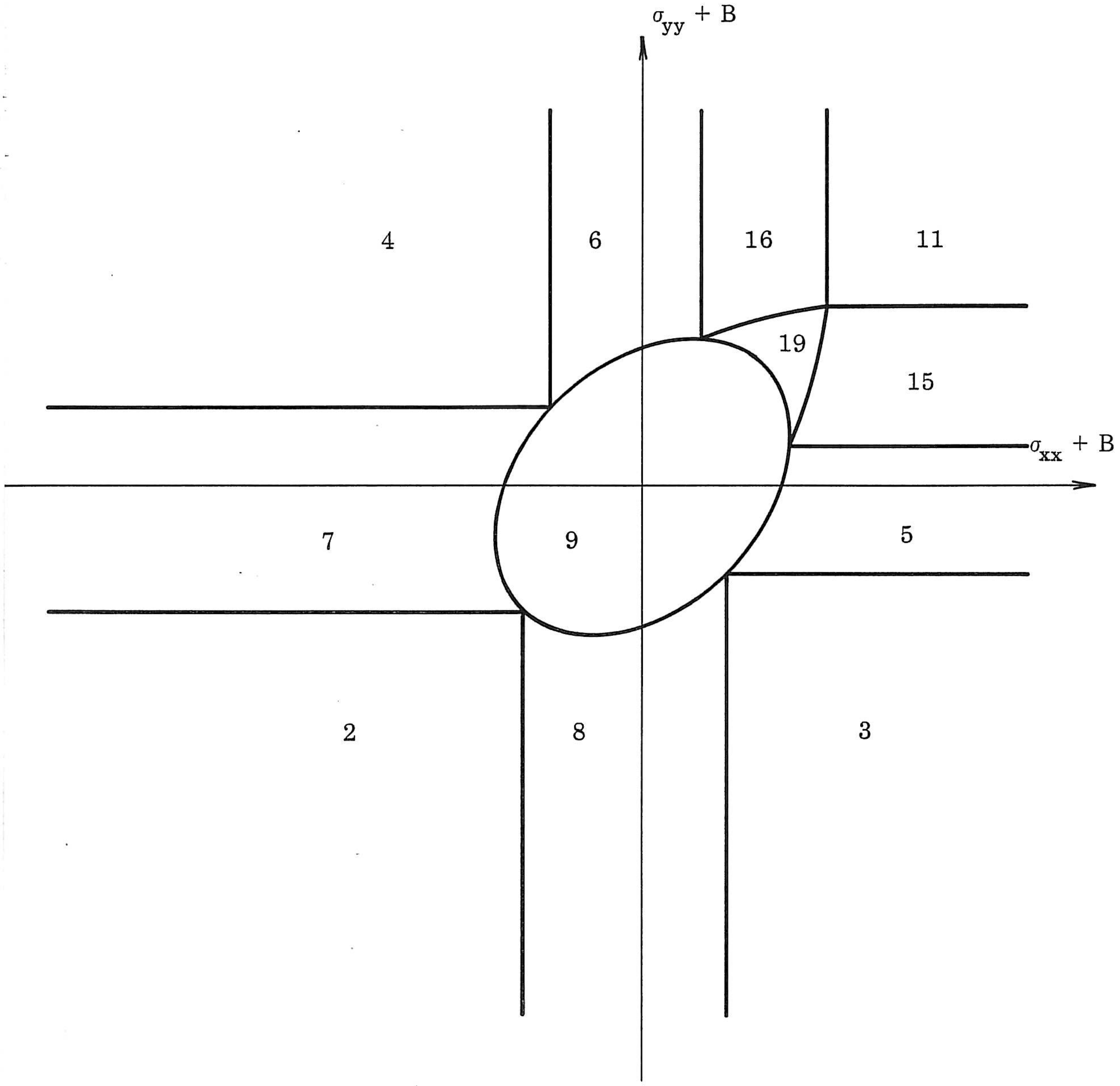
Figure 6.



A < 2Q

$Q/\sqrt{2} < \sigma_{xy} < 2Q^2/\sqrt{4Q^2 + A^2}$

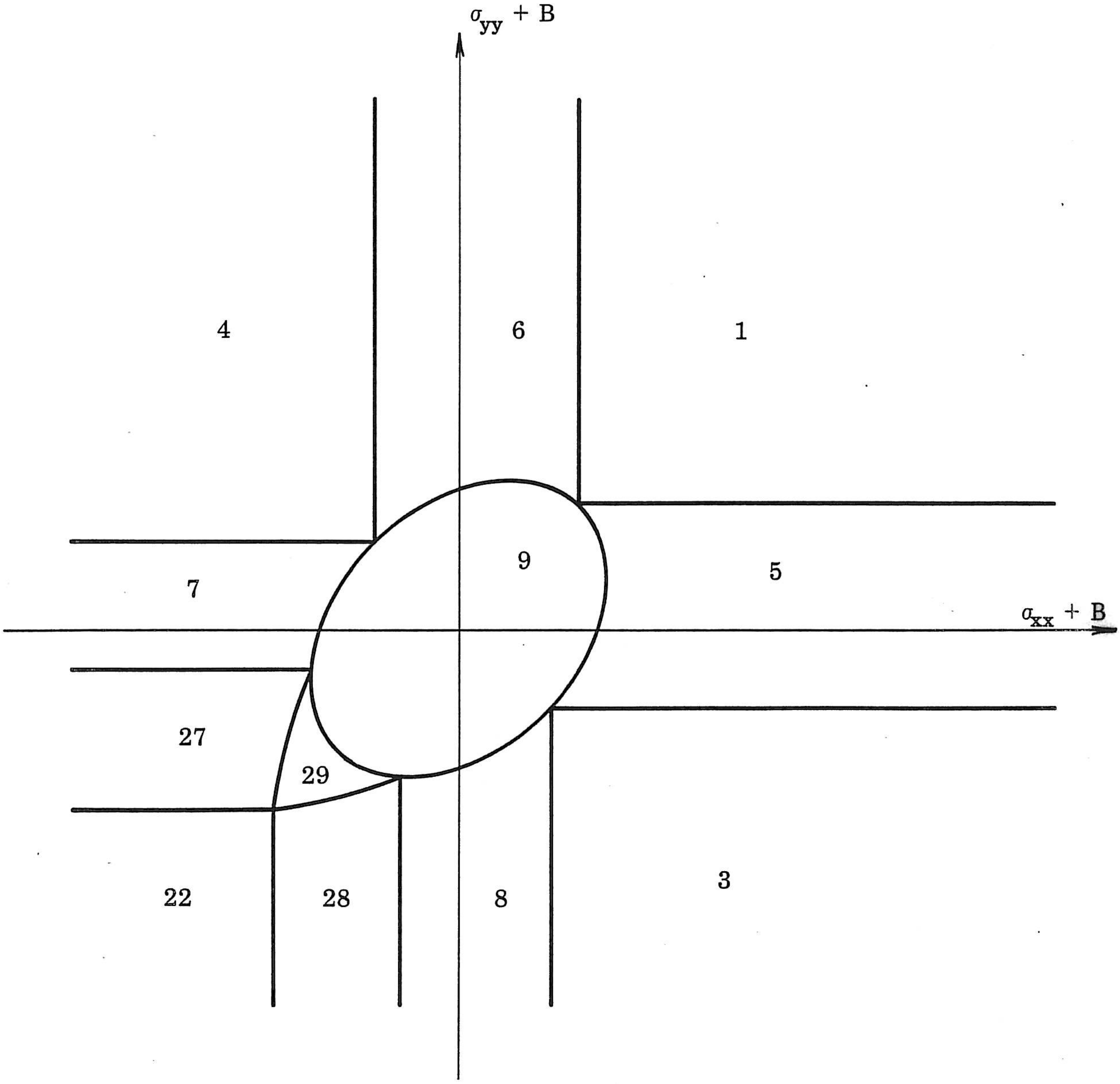
Figure 7.



$A < 2Q$

$0 < \sigma_{xy} < Q/\sqrt{2}$

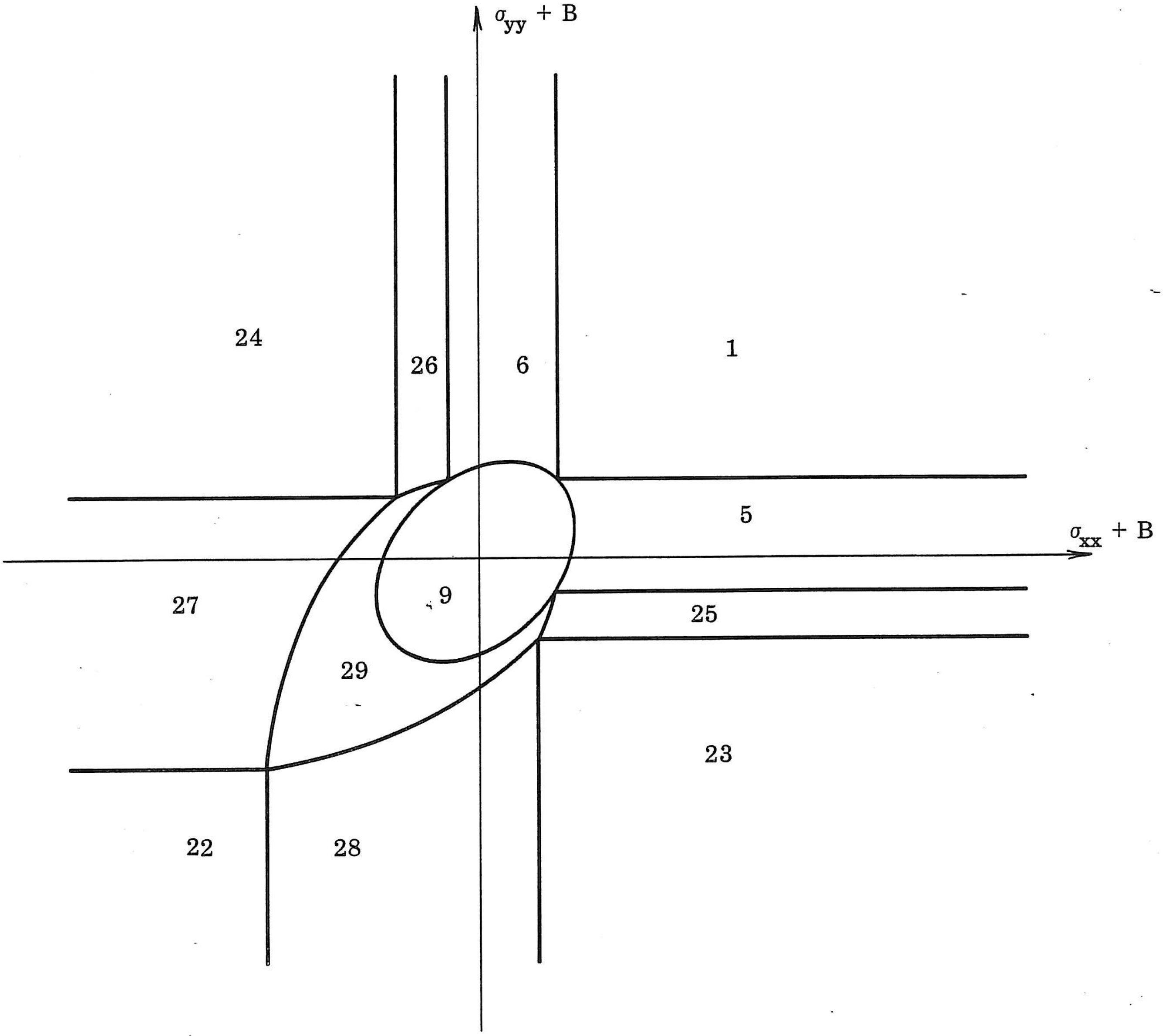
Figure 8.



$A < 2Q$

$-Q/\sqrt{2} < \sigma_{xy} < 0$

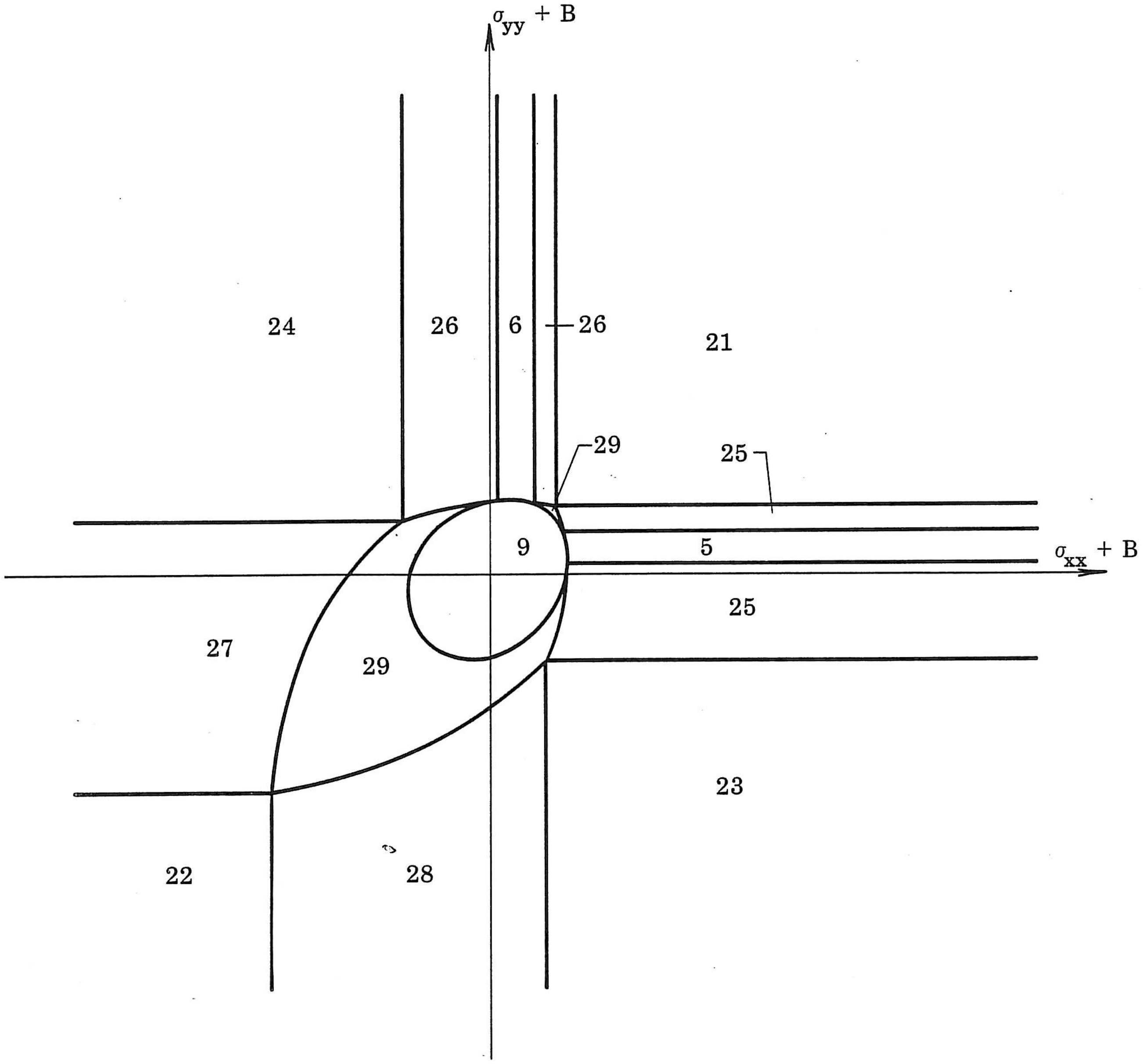
Figure 9.



A < 2Q

$-2Q^2/\sqrt{4Q^2 + A^2} < \sigma_{xy} < -Q/\sqrt{2}$

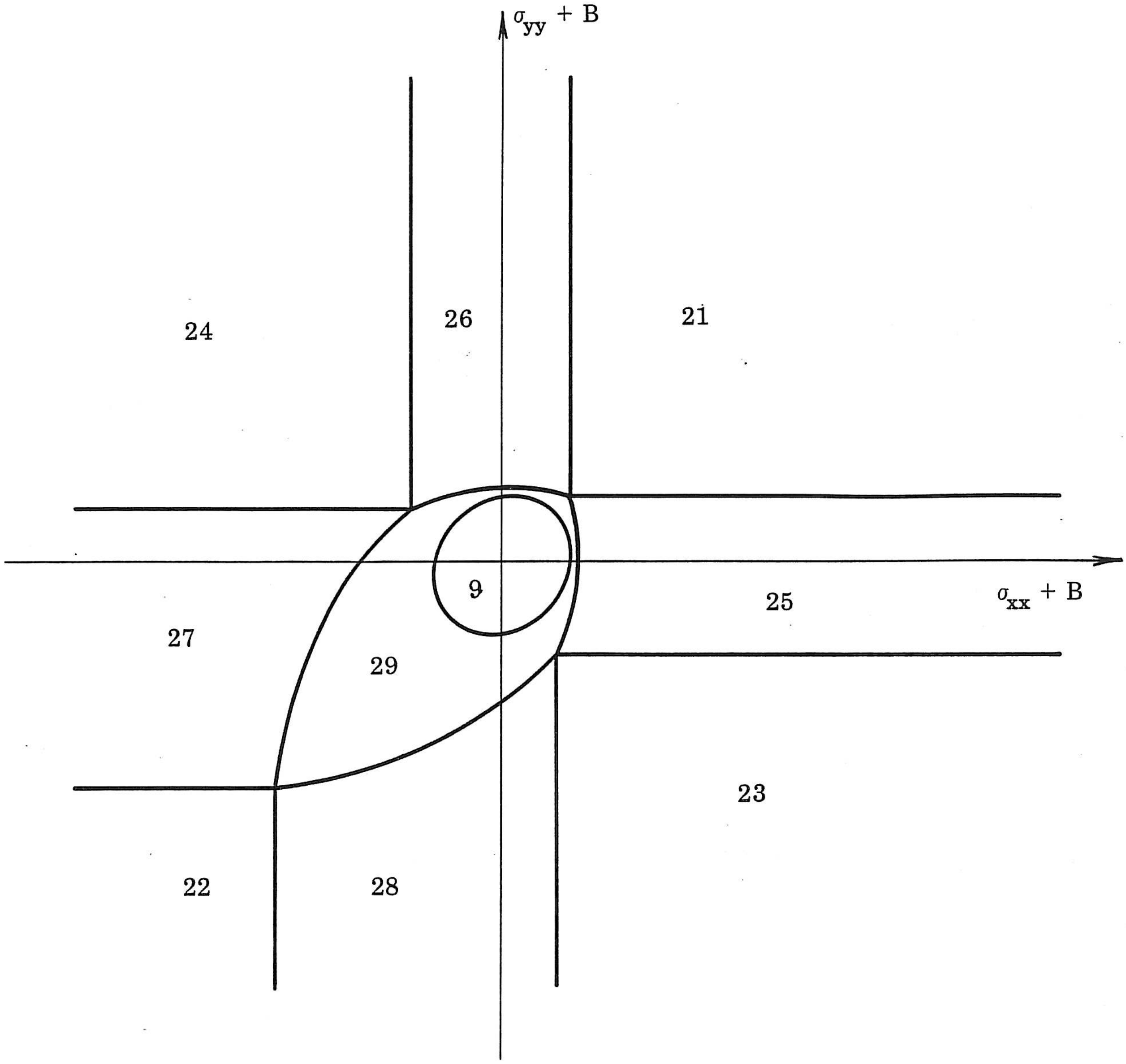
Figure 10.



$A < 2Q$

$$-Q\sqrt{4Q^2 + A^2} \,/\, \sqrt{2(2Q^2 + A^2)} < \sigma_{xy} < -2Q^2 \,/\, \sqrt{4Q^2 + A^2}$$

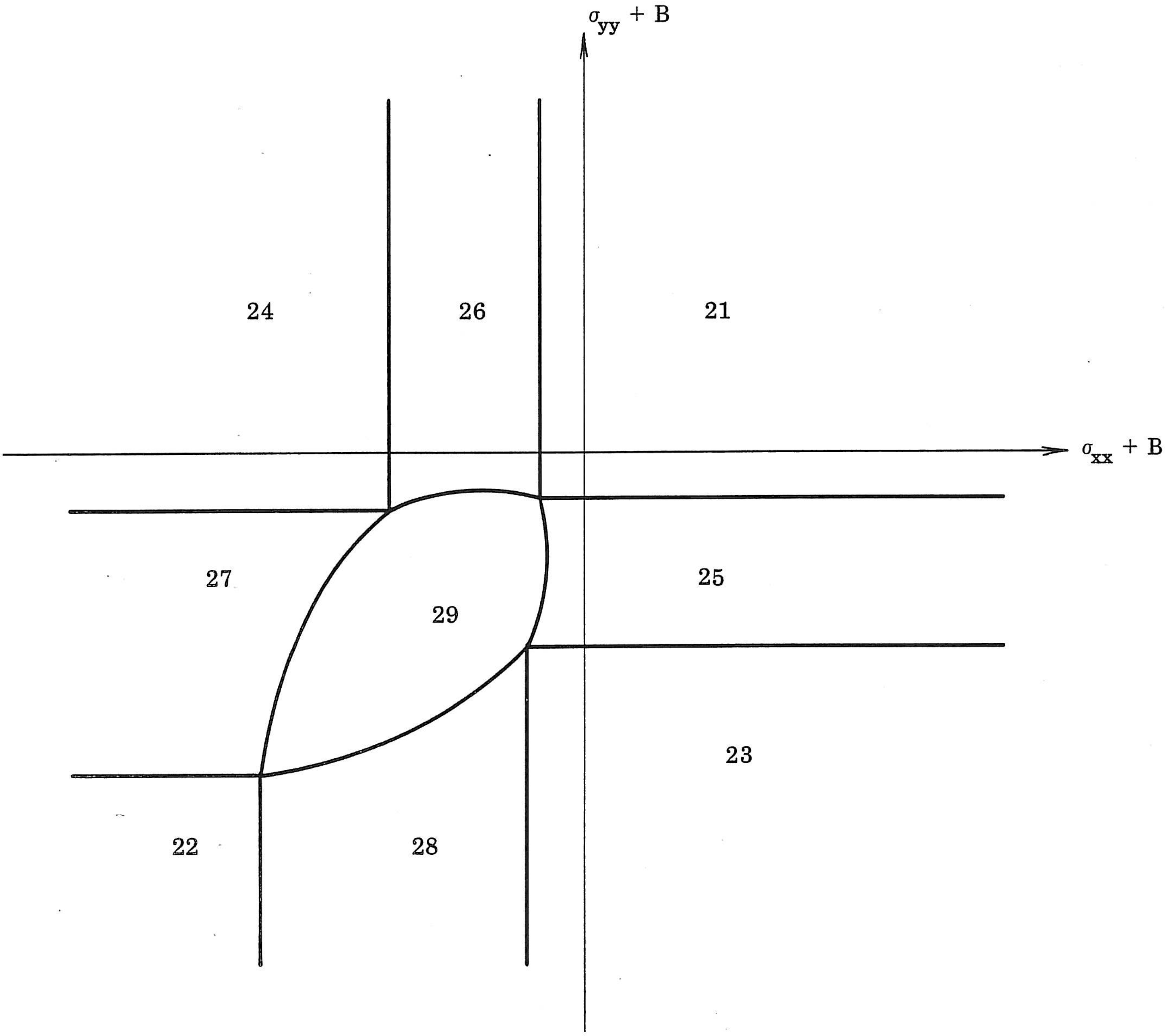
Figure 11.



$A < 2Q$

$-Q < \sigma_{xy} < -Q\sqrt{4Q^2 + A^2} \,/\, \sqrt{2(2Q^2 + A^2)}$

Figure 12.



$A < 2Q$

$\sigma_{xy} < -Q$

Figure 13.

Fields	Boundary
1 - 5	$\sigma_{yy} + B = A\sqrt{1-(\sigma_{xy}/Q)^2}$
1 - 6	$\sigma_{xx} + B = A\sqrt{1-(\sigma_{xy}/Q)^2}$
1 - 11	$\sigma_{xy} = 0$
1 - 21	$\sigma_{xy} = -2Q^2/\sqrt{4Q^2 + A^2}$
2 - 7	$\sigma_{yy} + B = -A\sqrt{1-(\sigma_{xy}/Q)^2}$
2 - 8	$\sigma_{xx} + B = -A\sqrt{1-(\sigma_{xy}/Q)^2}$
2 - 12	$\sigma_{xy} = 2Q^2/\sqrt{4Q^2 + A^2}$
2 - 22	$\sigma_{xy} = 0$
3 - 5	$\sigma_{yy} + B = -\sqrt{Q^2 - \sigma_{xy}^2}$
3 - 8	$\sigma_{xx} + B = \sqrt{Q^2 - \sigma_{xy}^2}$
3 -13	$\sigma_{xy} = Q/\sqrt{2}$
3 - 23	$\sigma_{xy} = -Q/\sqrt{2}$
4 - 6	$\sigma_{xx} + B = -\sqrt{Q^2 - \sigma_{xy}^2}$
4 - 7	$\sigma_{yy} + B = \sqrt{Q^2 - \sigma_{xy}^2}$
4 - 14	$\sigma_{xy} = Q/\sqrt{2}$
4 - 24	$\sigma_{xy} = -Q/\sqrt{2}$
5 - 15	$h_1 = 0$
5 - 25	$h_3 = 0$
6 - 16	$h_2 = 0$
6 - 26	$h_4 = 0$
7 - 17	$h_3 = 0$
7 - 27	$h_1 = 0$
8 - 18	$h_4 = 0$
8 - 28	$h_2 = 0$
9 - 5	$f = 0$
9 - 6	$f = 0$
9 - 7	$f = 0$
9 - 8	$f = 0$
9 - 19	$f = 0$
9 - 29	$f = 0$

Table 4 (continued)

Fields	Boundary
11 - 16	$\sigma_{yy} + B = \sigma_{xy} + A$
11 - 16	$\sigma_{xx} + B = \sigma_{xy} + A$
12 - 17	$\sigma_{yy} + B = \sigma_{xy} - (2Q^2 + A^2)/\sqrt{4Q^2 + A^2}$
12 - 18	$\sigma_{xx} + B = \sigma_{xy} - (2Q^2 + A^2)/\sqrt{4Q^2 + A^2}$
13 - 15	$\sigma_{yy} + B = \sigma_{xy} - \sqrt{2}\,Q$
13 - 18	$\sigma_{xx} + B = \sigma_{xy}$
14 - 16	$\sigma_{xx} + B = \sigma_{xy} - \sqrt{2}\,Q$
14 - 18	$\sigma_{yy} + B = \sigma_{xy}$
19 - 15	$g_1 = 0$
19 - 16	$g_2 = 0$
19 - 17	$g_3 = 0$
19 - 18	$g_4 = 0$
21 - 25	$\sigma_{yy} + B = \sigma_{xy} + (2Q^2 + A^2)/\sqrt{4Q^2 + A^2}$
21 - 26	$\sigma_{xx} + B = \sigma_{xy} + (2Q^2 + A^2)/\sqrt{4Q^2 + A^2}$
22 - 27	$\sigma_{yy} + B = \sigma_{xy} - A$
22 - 28	$\sigma_{xx} + B = \sigma_{xy} - A$
23 - 25	$\sigma_{yy} + B = \sigma_{xy}$
23 - 28	$\sigma_{xx} + B = \sigma_{xy} + \sqrt{2}\,Q$
24 - 26	$\sigma_{xx} + B = \sigma_{xy}$
24 - 27	$\sigma_{yy} + B = \sigma_{xy} + \sqrt{2}\,Q$
29 - 25	$g_3 = 0$
29 - 26	$g_4 = 0$
29 - 27	$g_1 = 0$
29 - 28	$g_2 = 0$

Table 4 (continued)

$$f(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = Q^2(\sigma_{xx} + \sigma_{yy} + 2B)^2 + A^2(\sigma_{xx} - \sigma_{yy})^2 + 4A^2\sigma_{xy}^2 - 4A^2Q^2$$

$$g_1(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = A^2(\sigma_{xx} - \sigma_{yy})^2 + 2Q^2(\sigma_{xx} + B - \sigma_{xy})^2 - 2A^2Q^2$$

$$g_2(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = A^2(\sigma_{xx} - \sigma_{yy})^2 + 2Q^2(\sigma_{yy} + B - \sigma_{xy})^2 - 2A^2Q^2$$

$$g_3(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = (A^2(\sigma_{yy} + B - \sigma_{xy}) - (2Q^2 + A^2)(\sigma_{xx} + B - \sigma_{xy}))^2 - 2Q^2(2Q^2 + A^2)(2Q^2 + A^2 - (\sigma_{yy} + B - \sigma_{xy})^2)$$

$$g_4(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = (A^2(\sigma_{xx} + B - \sigma_{xy}) - (2Q^2 + A^2)(\sigma_{yy} + B - \sigma_{xy}))^2 - 2Q^2(2Q^2 + A^2)(2Q^2 + A^2 - (\sigma_{xx} + B - \sigma_{xy})^2)$$

$$h_1(\sigma_{yy}, \sigma_{xy}) = Q^2(\sigma_{yy} + B + \sigma_{xy})^2 - A^2(Q^2 - 2\sigma_{xy}^2)$$

$$h_2(\sigma_{xx}, \sigma_{xy}) = Q^2(\sigma_{xx} + B + \sigma_{xy})^2 - A^2(Q^2 - 2\sigma_{xy}^2)$$

$$h_3(\sigma_{yy}, \sigma_{xy}) = Q^2((4Q^2 + A^2)(\sigma_{yy} + B) + A^2\sigma_{xy})^2 - (2Q^2 + A^2)(Q^2(4Q^2 + A^2) - 2(2Q^2 + A^2)\sigma_{xy}^2)$$

$$h_4(\sigma_{xx}, \sigma_{xy}) = Q^2((4Q^2 + A^2)(\sigma_{xx} + B) + A^2\sigma_{xy})^2 - (2Q^2 + A^2)(Q^2(4Q^2 + A^2) - 2(2Q^2 + A^2)\sigma_{xy}^2)$$

Table 4.

Field	$\varphi_\theta t_\theta$	$\varphi_x t_x$	$\varphi_y t_y$	m_{xy}	$m_{xx} + B$	$m_{yy} + B$
1	0	$\varphi_x t_{tx}$	$\varphi_y t_{ty}$	σ_{xy}	$+A\sqrt{1-(\sigma_{xy}/Q)^2}$	$+A\sqrt{1-(\sigma_{xy}/Q)^2}$
2	0	$-\varphi_x t_{cx}$	$-\varphi_y t_{cy}$	σ_{xy}	$-A\sqrt{1-(\sigma_{xy}/Q)^2}$	$-A\sqrt{1-(\sigma_{xy}/Q)^2}$
3	0	$\varphi_x t_{tx}$	$-\varphi_y t_{cy}$	σ_{xy}	$+Q\sqrt{1-(\sigma_{xy}/Q)^2}$	$-Q\sqrt{1-(\sigma_{xy}/Q)^2}$
4	0	$-\varphi_x t_{cx}$	$\varphi_y t_{ty}$	σ_{xy}	$-Q\sqrt{1-(\sigma_{xy}/Q)^2}$	$+Q\sqrt{1-(\sigma_{xy}/Q)^2}$
5	0	$\varphi_x t_{tx}$	0	σ_{xy}	$K_1 + L_1$	$\sigma_{yy} + B$
6	0	0	$\varphi_y t_{ty}$	σ_{xy}	$\sigma_{xx} + B$	$K_2 + L_2$
7	0	$-\varphi_x t_{cx}$	0	σ_{xy}	$K_1 - L_1$	$\sigma_{yy} + B$
8	0	0	$-\varphi_y t_{cy}$	σ_{xy}	$\sigma_{xx} + B$	$K_2 - L_2$
9	0	0	0	σ_{xy}	$\sigma_{xx} + B$	$\sigma_{yy} + B$
11	$\varphi_\theta t_{t\theta}$	$\varphi_x t_{tx}$	$\varphi_y t_{ty}$	0	A	A
12	$\varphi_\theta t_{t\theta}$	$-\varphi_x t_{cx}$	$-\varphi_y t_{cy}$	$2Q^2/\sqrt{4Q^2+A^2}$	$-A^2 m_{xy}/2Q^2$	$-A^2 m_{xy}/2Q^2$
13	$\varphi_\theta t_{t\theta}$	$\varphi_x t_{tx}$	$-\varphi_y t_{cy}$	$Q/\sqrt{2}$	$+m_{xy}$	$-m_{xy}$
14	$\varphi_\theta t_{t\theta}$	$-\varphi_x t_{cx}$	$\varphi_y t_{ty}$	$Q/\sqrt{2}$	$-m_{xy}$	$+m_{xy}$
15	$\varphi_\theta t_{t\theta}$	$\varphi_x t_{tx}$	0	$M_1 + N_1$	$3m_{xy}-\sigma_{xy}+\sigma_{yy}+B$	$m_{xy}-\sigma_{xy}+\sigma_{yy}+B$
16	$\varphi_\theta t_{t\theta}$	0	$\varphi_y t_{ty}$	$M_2 + N_2$	$m_{xy}-\sigma_{xy}+\sigma_{xx}+B$	$3m_{xy}-\sigma_{xy}+\sigma_{xx}+B$
17	$\varphi_\theta t_{t\theta}$	$-\varphi_x t_{cx}$	0	$M_1 + N_3$	$-m_{xy}-P_1$	$m_{xy}-\sigma_{xy}+\sigma_{yy}+B$
18	$\varphi_\theta t_{t\theta}$	0	$-\varphi_y t_{cy}$	$M_2 + N_4$	$m_{xy}-\sigma_{xy}+\sigma_{xx}+B$	$-m_{xy}-P_2$
19	$\varphi_\theta t_{t\theta}$	0	0	$R_1 + R_2$	$m_{xy}-\sigma_{xy}+\sigma_{xx}+B$	$m_{xy}-\sigma_{xy}+\sigma_{yy}+B$
21	$-\varphi_\theta t_{c\theta}$	$\varphi_x t_{tx}$	$\varphi_y t_{ty}$	$-2Q^2/\sqrt{4Q^2+A^2}$	$-A^2 m_{xy}/2Q^2$	$-A^2 m_{xy}/2Q^2$
22	$-\varphi_\theta t_{c\theta}$	$-\varphi_x t_{cx}$	$-\varphi_y t_{cy}$	0	$-A$	$-A$
23	$-\varphi_\theta t_{c\theta}$	$\varphi_x t_{tx}$	$-\varphi_y t_{cy}$	$-Q/\sqrt{2}$	$+m_{xy}$	$-m_{xy}$
24	$-\varphi_\theta t_{c\theta}$	$-\varphi_x t_{cx}$	$\varphi_y t_{ty}$	$-Q/\sqrt{2}$	$-m_{xy}$	$+m_{xy}$
25	$-\varphi_\theta t_{c\theta}$	$\varphi_x t_{tx}$	0	$M_1 - N_3$	$-m_{xy}-P_1$	$m_{xy}-\sigma_{xy}+\sigma_{yy}+B$
26	$-\varphi_\theta t_{c\theta}$	0	$\varphi_y t_{ty}$	$M_2 - N_4$	$m_{xy}-\sigma_{xy}+\sigma_{xx}+B$	$-m_{xy}-P_2$
27	$-\varphi_\theta t_{c\theta}$	$-\varphi_x t_{cx}$	0	$M_1 - N_1$	$3m_{xy}-\sigma_{xy}+\sigma_{yy}+B$	$m_{xy}-\sigma_{xy}+\sigma_{yy}+B$
28	$-\varphi_\theta t_{c\theta}$	0	$-\varphi_y t_{cy}$	$M_2 - N_2$	$m_{xy}-\sigma_{xy}+\sigma_{xx}+B$	$3m_{xy}-\sigma_{xy}+\sigma_{xx}+B$
29	$-\varphi_\theta t_{c\theta}$	0	0	$R_1 - R_2$	$m_{xy}-\sigma_{xy}+\sigma_{xx}+B$	$m_{xy}-\sigma_{xy}+\sigma_{yy}+B$

Table 5.

$$K_1 = (A^2 - Q^2)(\sigma_{yy} + B) / (A^2 + Q^2)$$

$$K_2 = (A^2 - Q^2)(\sigma_{xx} + B) / (A^2 + Q^2)$$

$$L_1 = 2A\sqrt{(A^2 + Q^2)(Q^2 - \sigma_{xy}^2) - Q^2(\sigma_{yy} + B)^2} / (A^2 + Q^2)$$

$$L_2 = 2A\sqrt{(A^2 + Q^2)(Q^2 - \sigma_{xy}^2) - Q^2(\sigma_{xx} + B)^2} / (A^2 + Q^2)$$

$$M_1 = -Q^2(\sigma_{yy} + B - \sigma_{xy}) / (2Q^2 + A^2)$$

$$M_2 = -Q^2(\sigma_{xx} + B - \sigma_{xy}) / (2Q^2 + A^2)$$

$$N_1 = AQ\sqrt{2(2Q^2 + A^2 - (\sigma_{yy} + B - \sigma_{xy})^2)} / 2(2Q^2 + A^2)$$

$$N_2 = AQ\sqrt{2(2Q^2 + A^2 - (\sigma_{xx} + B - \sigma_{xy})^2)} / 2(2Q^2 + A^2)$$

$$N_3 = Q\sqrt{2(2Q^2 + A^2)(2Q^2 + A^2 - (\sigma_{yy} + B - \sigma_{xy})^2)} / 2(2Q^2 + A^2)$$

$$N_4 = Q\sqrt{2(2Q^2 + A^2)(2Q^2 + A^2 - (\sigma_{xx} + B - \sigma_{xy})^2)} / 2(2Q^2 + A^2)$$

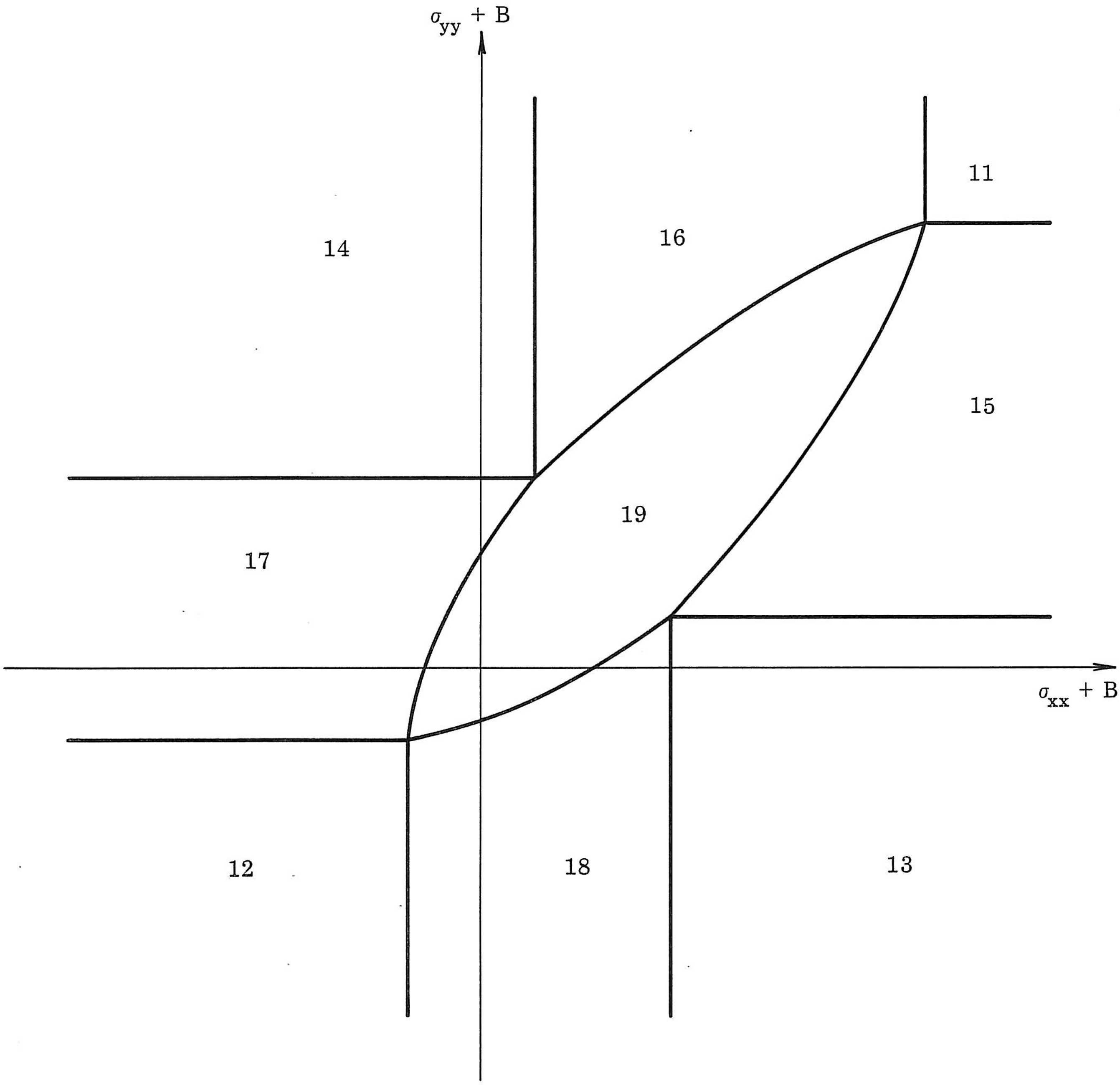
$$P_1 = (2Q^2 - A^2)(\sigma_{yy} + B - \sigma_{xy}) / (2Q^2 + A^2)$$

$$P_2 = (2Q^2 - A^2)(\sigma_{xx} + B - \sigma_{xy}) / (2Q^2 + A^2)$$

$$R_1 = -Q^2(\sigma_{xx} + \sigma_{yy} + 2B - 2\sigma_{xy}) / 2(Q^2 + A^2)$$

$$R_2 = A\sqrt{4Q^2(Q^2 + A^2) - Q^2(\sigma_{xx} + \sigma_{yy} + 2B - 2\sigma_{xy})^2 - (Q^2 + A^2)(\sigma_{xx} - \sigma_{yy})^2} / 2(Q^2 + A^2)$$

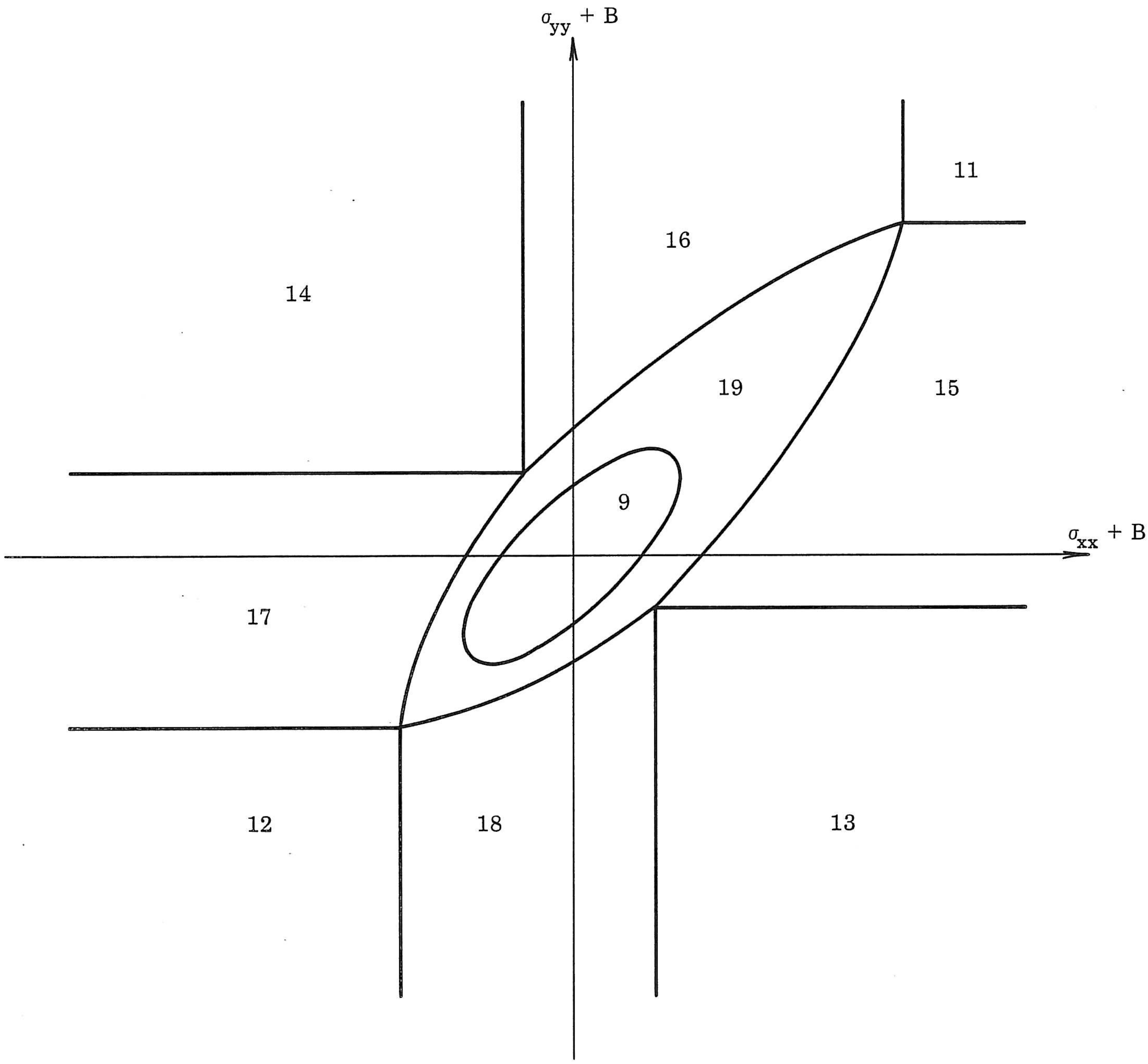
Table 5.



$2Q < A$

$Q < \sigma_{xy}$

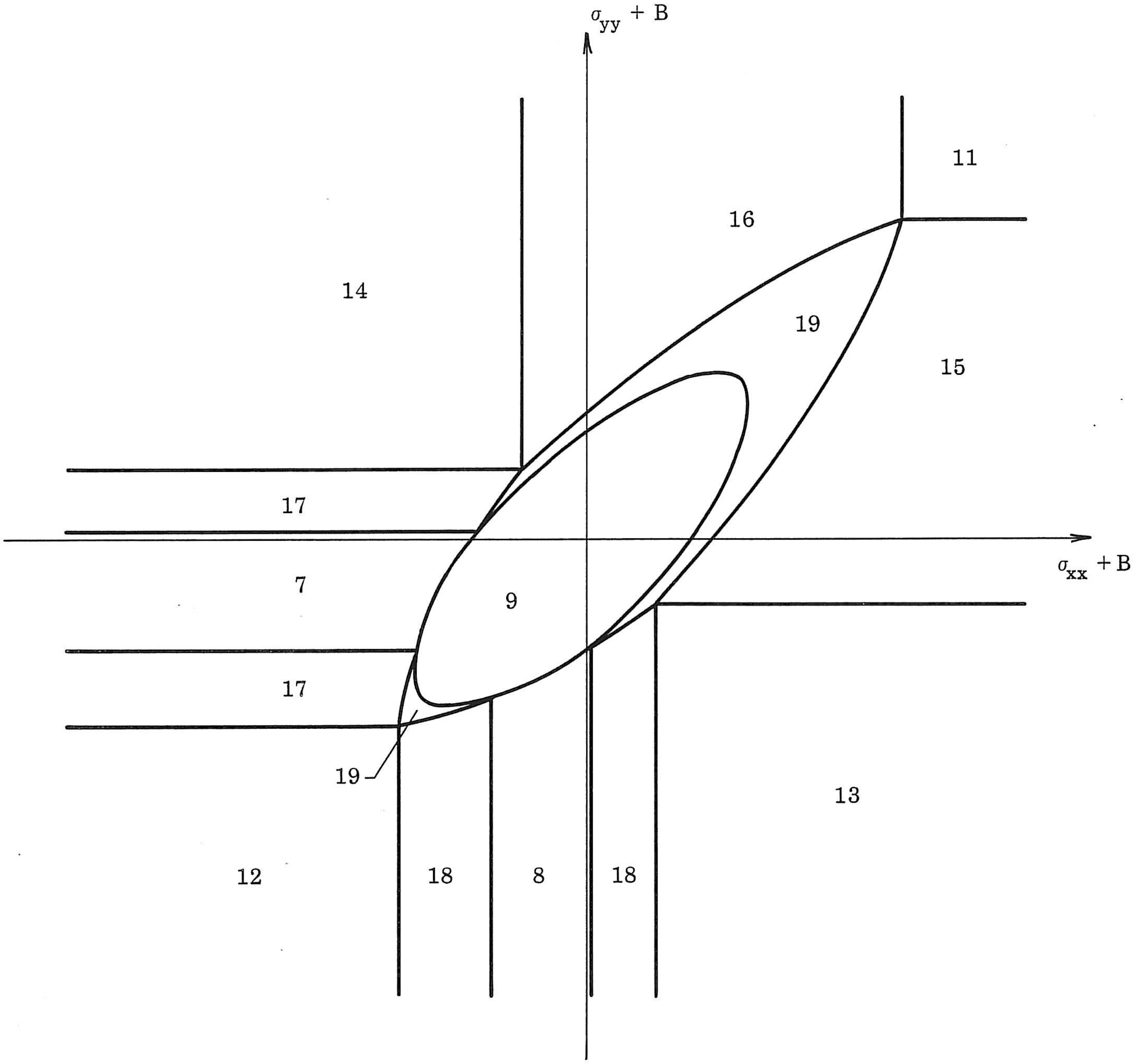
Figure 14



$2Q < A$

$Q\sqrt{4Q^2 + A^2} \,/\, \sqrt{2(2Q^2 + A^2)} < \sigma_{xy} < Q$

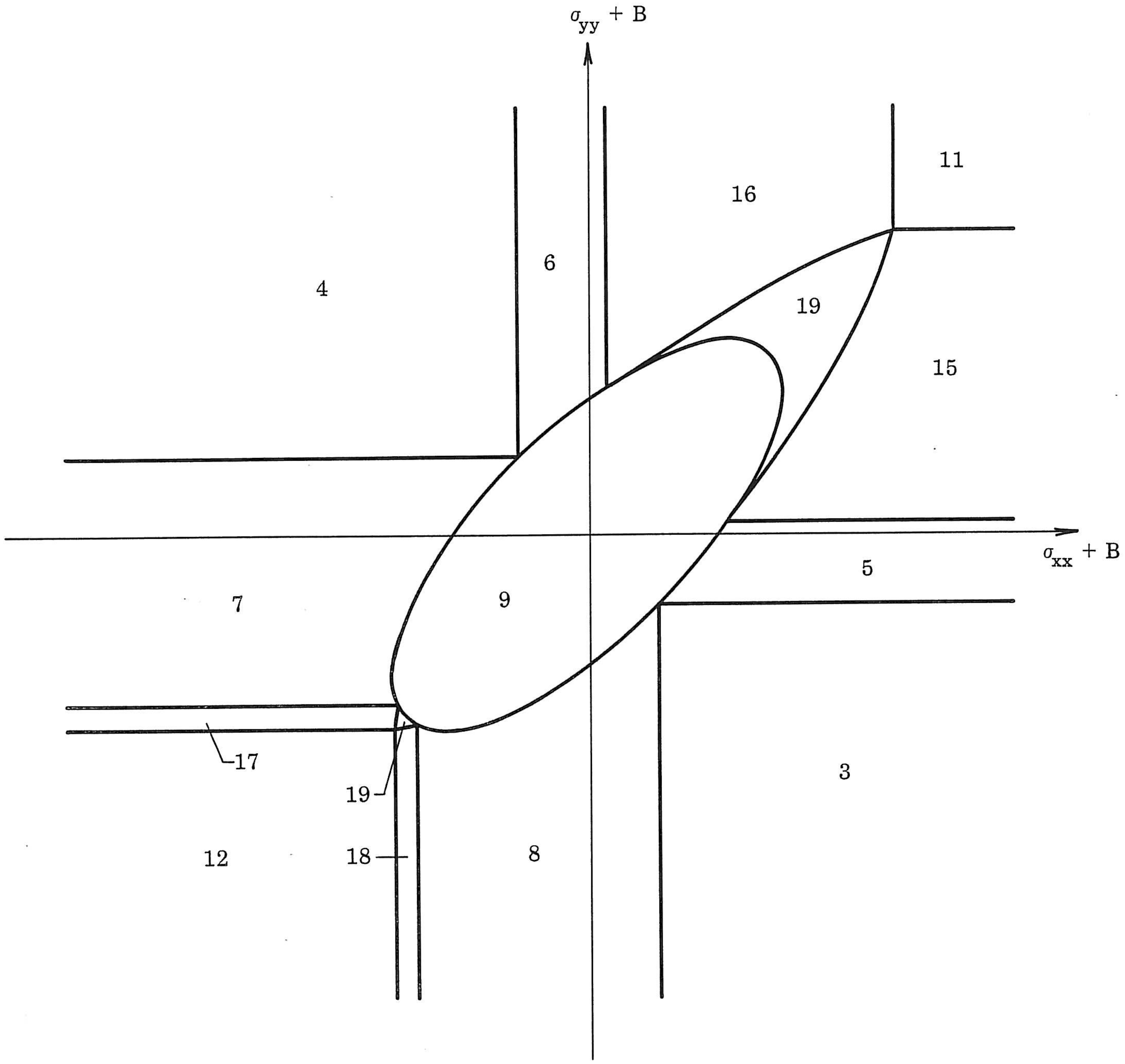
Figure 15



2Q < A

$Q / \sqrt{2} < \sigma_{xy} < Q \sqrt{4Q^2 + A^2} / \sqrt{2(2Q^2 + A^2)}$

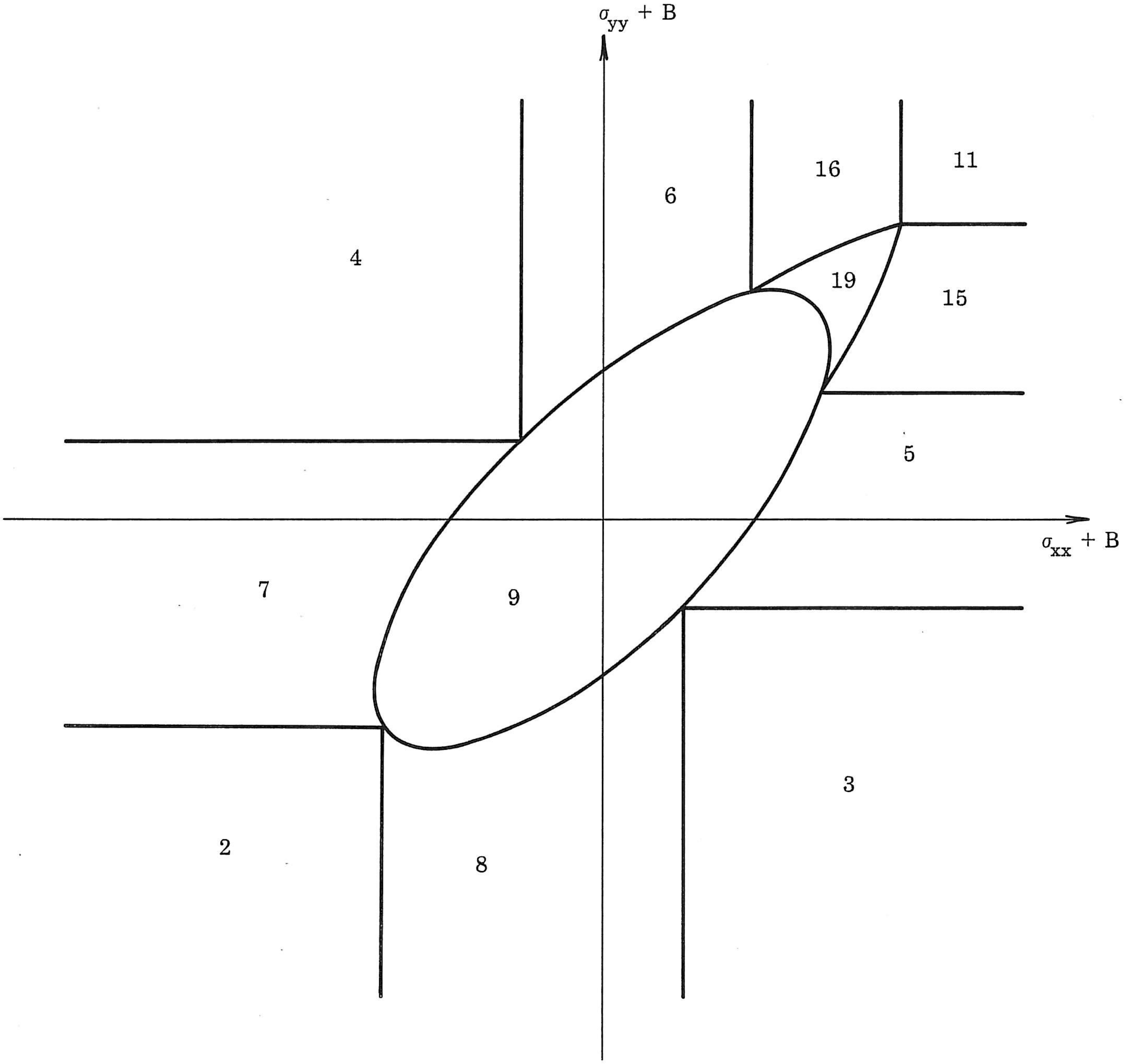
Figure 16



2Q < A

$2Q^2 / \sqrt{4Q^2 + A^2} < \sigma_{xy} < Q / \sqrt{2}$

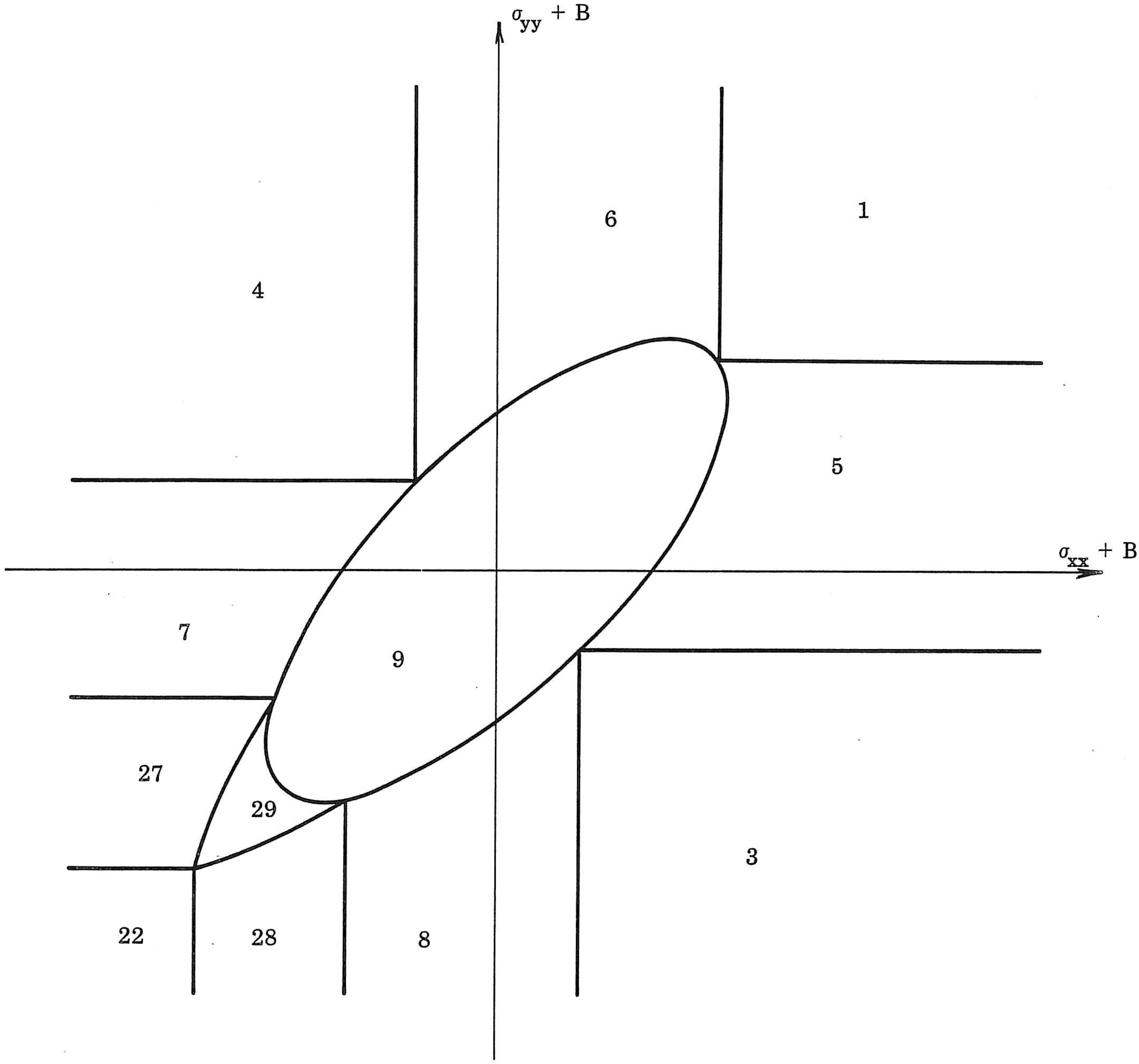
Figure 17



$2Q < A$

$0 < \sigma_{xy} < 2Q^2 / \sqrt{4Q^2 + A^2}$

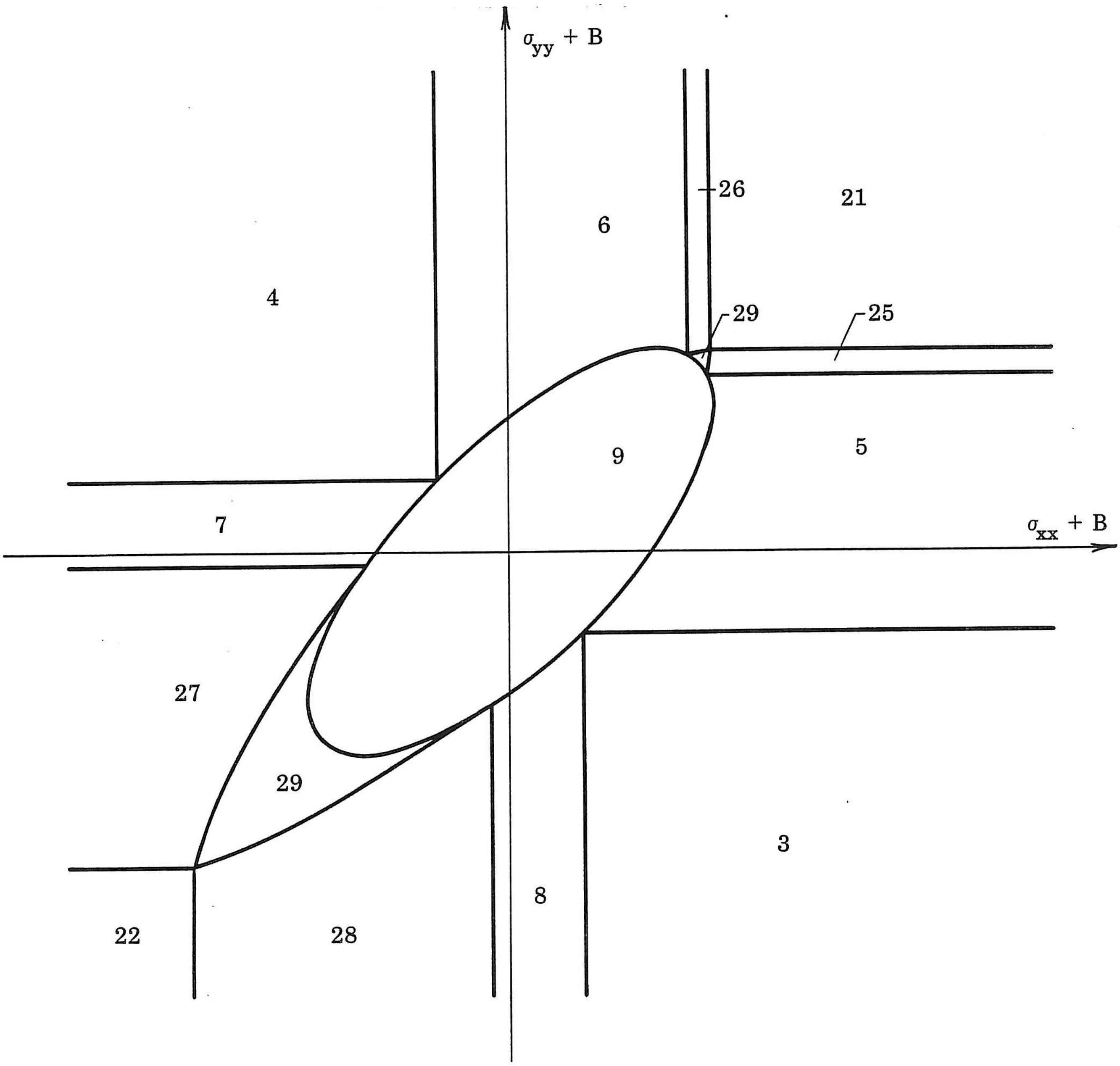
Figure 18.



$2Q < A$

$-2Q^2/\sqrt{4Q^2 + A^2} < \sigma_{xy} < 0$

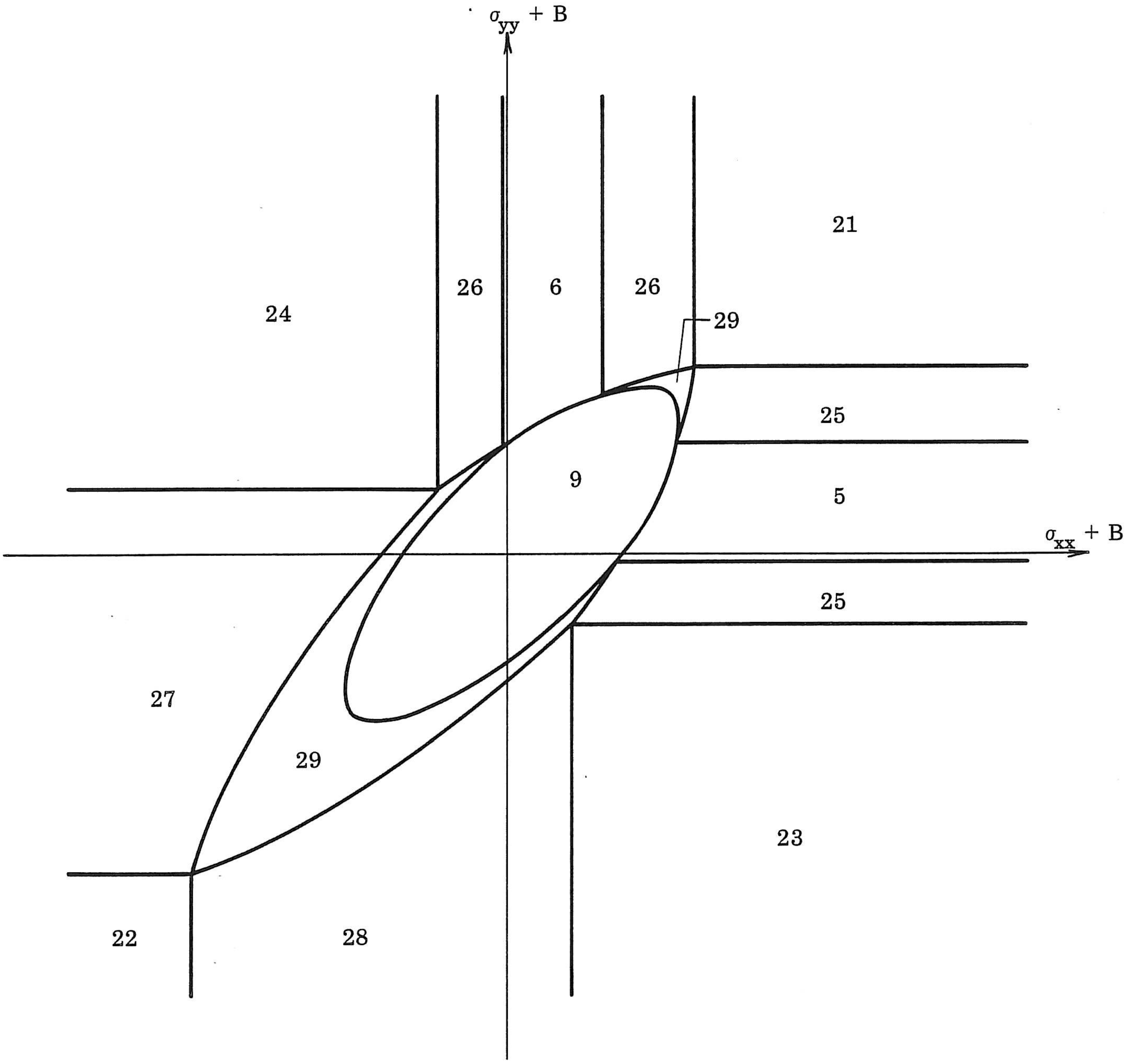
Figure 19.



$2Q < A$

$-Q/\sqrt{2} < \sigma_{xy} < -2Q^2/\sqrt{4Q^2 + A^2}$

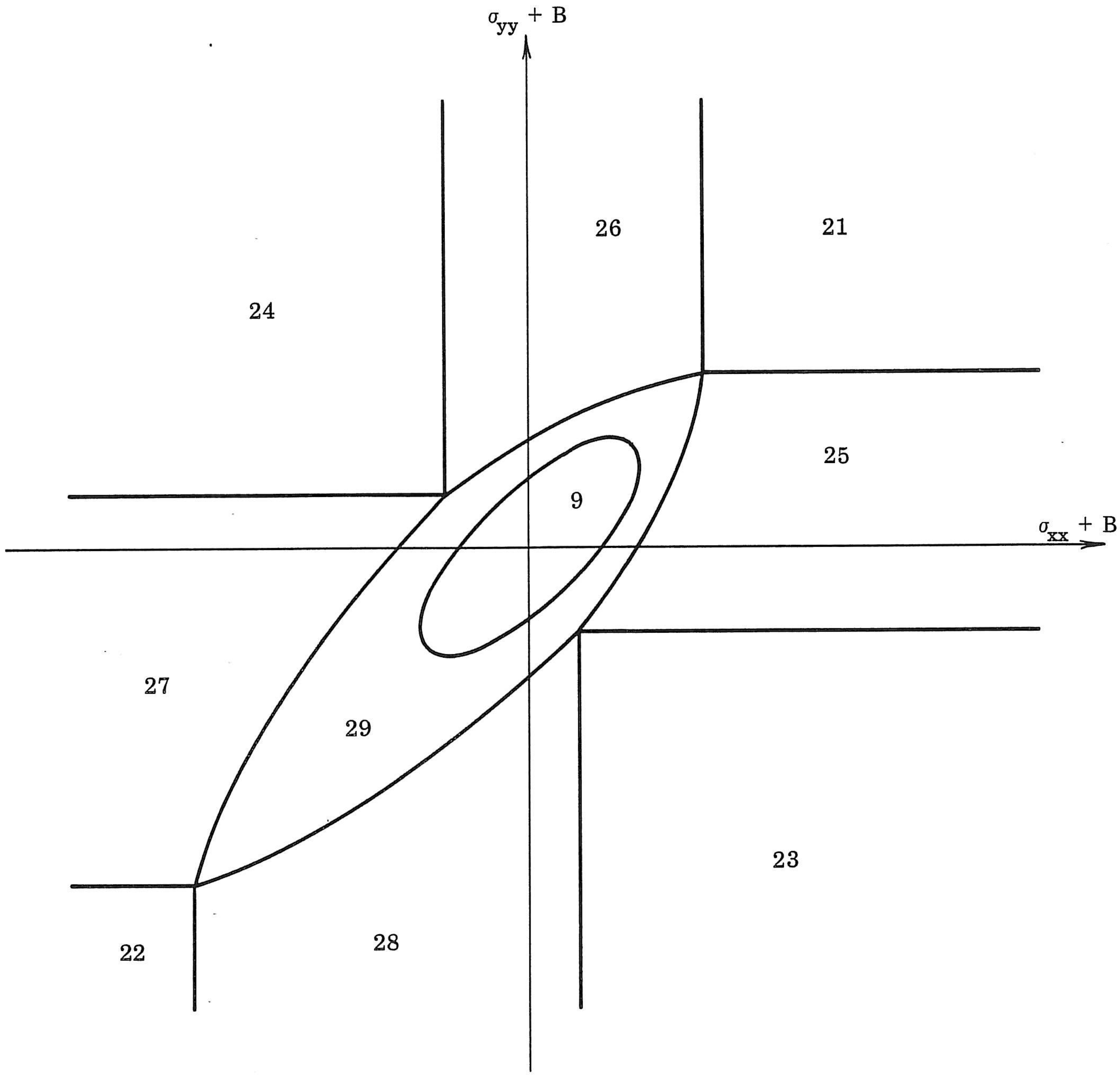
Figure 20.



$2Q < A$

$-Q\sqrt{4Q^2 + A^2} / \sqrt{2(2Q^2 + A^2)} < \sigma_{xy} < -Q / \sqrt{2}$

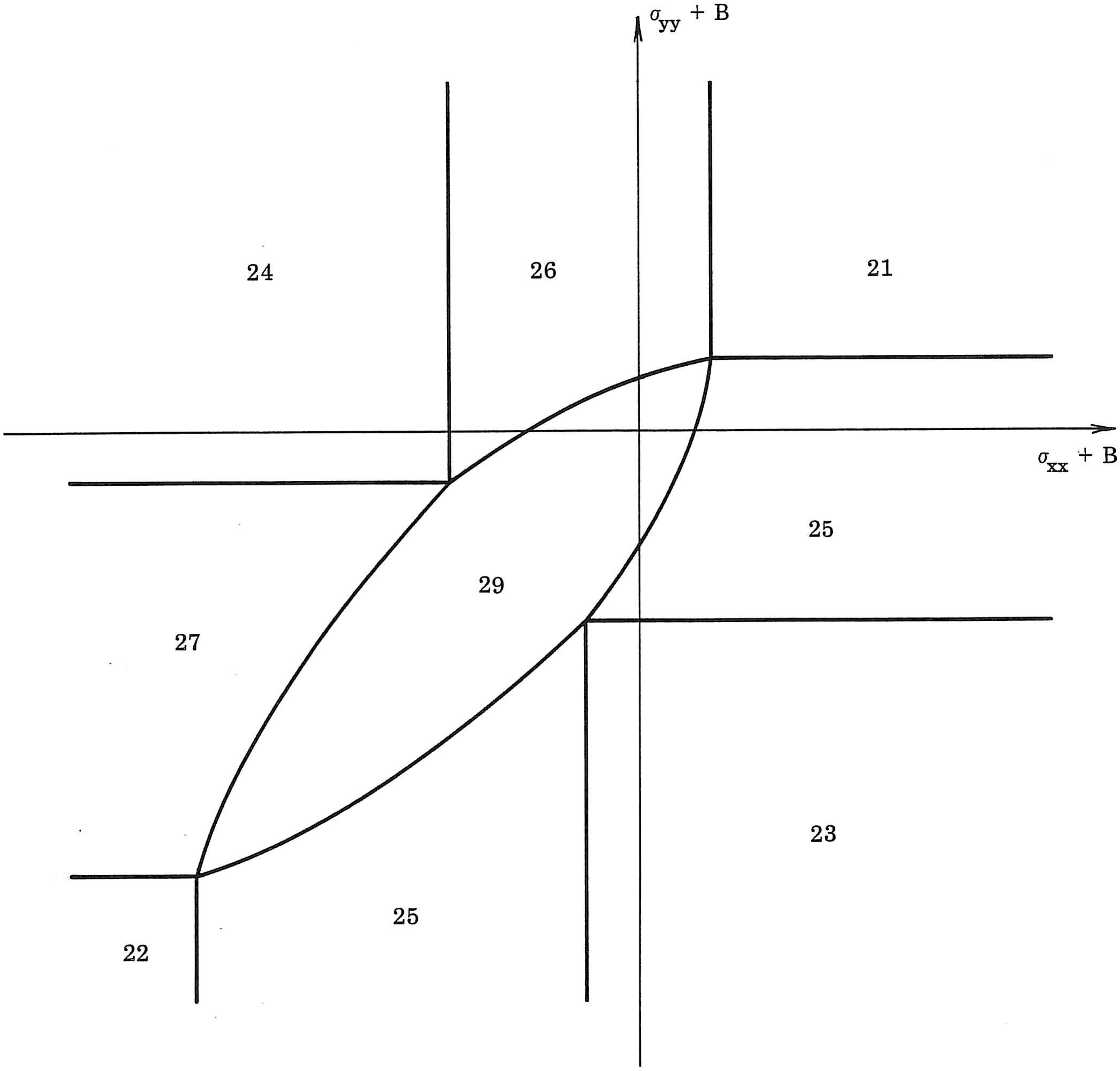
Figure 21.



$2Q < A$

$-Q < \sigma_{xy} < -Q\sqrt{4Q^2 + A^2} / \sqrt{2(2Q^2 + A^2)}$

Figure 22.



$2Q < A$

$\sigma_{xy} < -Q$

Figure 23.

5. CONCLUDING REMARKS

Minimum reinforcement is determined for composites loaded in plane stress with known stresses. The composites are isotropic matrices reinforced in either two orthogonal directions chosen in advance or in three directions, two orthogonal and a third inclined at 45^{o} to the two others.

It is assumed that both matrix and reinforcement can carry failure stress at the same time. If the ultimate strain of the matrix is greater than the ultimate strain of the reinforcement this should cause no problem. If the ultimate strain of the matrix, however, is less than the ultimate strain of the reinforcement, failure stress cannot occur in both materials at the same time. This problem often arises in connection with tensile stress and a simple and effective way to overcome the description of this phenomenon, is to supplement the matrix failure criterion with the principal stress criterion with no tensile strength i.e. $f = -II_{\sigma} = -\sigma_{xx}\sigma_{yy} + \sigma_{xy}^2 = 0$. The result is that all tensile stresses are carried by the reinforcement alone while shear and compressive stresses are carried by both reinforcement and matrix.

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